Linear differential equations

2. Second order equations with constant coefficients

$$a y'' + b y' + c y = 0$$

We need to find <u>2</u> linearly independent solutions, $y_1(x)$ & $y_2(x)$.

Try $y(x) = e^{\lambda x}$. Then $y'(x) = \lambda e^{\lambda x}$, $y''(x) = \lambda^2 e^{\lambda x}$.
Substitute into ODE:
$$a \lambda^2 e^{\lambda x} + b \lambda e^{\lambda x} + c e^{\lambda x} = 0$$
i.e. $e^{\lambda x} (a\lambda^2 + b\lambda + c) = 0$

Since $e^{\lambda x} \neq 0$ all the time, we have
$$a\lambda^2 + b\lambda + c = 0.$$

So finding solutions amounts to solving the <u>characteristic equation</u> $a\lambda^2 + b\lambda + c = 0$.

<u>Example 1:</u> $y'' + y' - 2y = 0$

Characteristic equation: $\lambda^2 + \lambda - 2 = 0 = (\lambda - 1)(\lambda + 2)$
Two linearly independent solutions are:
$$y_1(x) = e^{x} \quad \& \quad y_2(x) = e^{-2x}$$
General solution:
$$y(x) = C_1 e^{x} + C_2 e^{-2x}$$
(linear combinat of y_1 & y_2)

Impose initial conditions:

$$1 = y(0) = C_1 + C_2$$

$$y'(x) = C_1 e^x - 2C_2 e^{-2x}$$

$$0 = y'(0) = C_1 - 2C_2 \Rightarrow C_1 = 2C_2$$

Then $1 = C_1 + 2C_2 = 3C_2 \Rightarrow C_2 = \frac{1}{3}$

and $C_1 = 2C_2 = \frac{2}{3}$

So $\boxed{y(x) = \frac{2}{3} e^x + \frac{1}{3} e^{-2x}}$.

Example 2: $y'' + 6y' + 25y = 0$

$$\lambda^2 + 6\lambda + 25 = 0$$

$$\text{"}b^2 - 4ac\text{"} = 36 - 4 \cdot 25 = 36 - 100 < 0 \;!!$$