Linear differential equations with constant coefficients (continued)

Example 2: $y'' + 6y' + 25y = 0 \qquad y(0) = 0;\ y'(0) = 1.$

Characteristic equation: $d^2 + 6d + 25 = 0$

i.e. $(d+3)^2 - 9 + 25 = 0$ i.e. $(d+3)^2 + 16 = 0$

i.e. $(d+3)^2 = -16 = i^2\,16 = (i4)^2$

i.e.

$$d + 3 = \pm 4i \qquad \text{i.e.} \qquad d = -3 \pm 4i$$

Alternatively, use the quadratic formula:

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-6 \pm \sqrt{36 - 4 \cdot 25}}{2} = \frac{-6 \pm \sqrt{-8^2}}{2}$$

$$= \frac{-6 \pm 8i}{2} = -3 \pm 4i \quad \checkmark$$

Since we have 2 roots, we have 2 linearly independent solutions:

$$y_1(x) = e^{(-3+4i)x} \quad \text{and} \quad y_2(x) = e^{(-3-4i)x}$$

The general solution is of the form:

$$y_h(x) = A\,e^{(-3+4i)x} + B\,e^{(-3-4i)x}$$

Since we want y_h to be real, A & B have to be complex.

What is $e^{(-3+4i)x} = e^{-3x} e^{4ix}$
$= e^{-3x} \left(\cos(4x) + i \sin(4x) \right)$

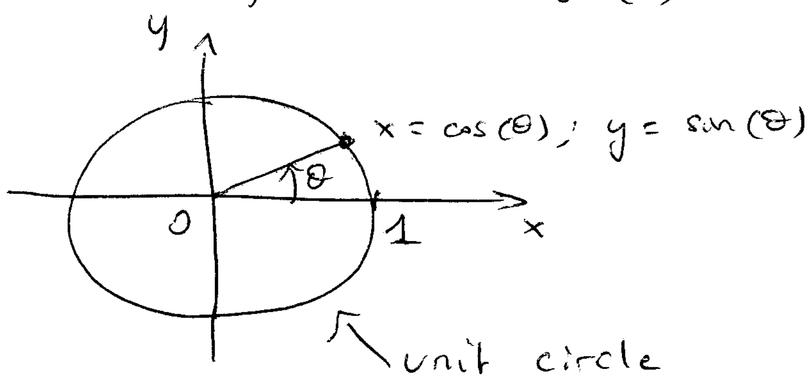
Recall Euler's formula: $e^{i\theta} = \cos(\theta) + i \sin(\theta)$

To check (or to know where this comes from):
Start with the series expansion of e^x:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2} + \frac{(i\theta)^3}{3!} + \dots$$

do the algebra ($i^2 = -1$)

$$= \underbrace{\cos(\theta)}_{\text{i.e. series of } \cos(\theta)} + i \underbrace{\sin(\theta)}_{\text{series of } \sin(\theta)}$$



$e^{i\theta} = 1 \left(\cos(\theta) + i \sin(\theta) \right).$

Now go back to the expression for $y_h(x)$:

$$y_h(x) = A e^{(-3+4i)x} + B e^{(-3-4i)x}$$
$$= A e^{-3x} \left(\cos(4x) + i \sin(4x) \right) + B e^{-3x} \left(\cos(-4x) + i \sin(-4x) \right)$$
$$= A e^{-3x} \left(\overbrace{\cos(4x) + i \sin(4x)}^{e^{4ix}} \right) + B e^{-3x} \left(\underbrace{\cos(4x) - i \sin(4x)}_{e^{-4ix}} \right)$$

Note that e^{4ix} is the complex conjugate of e^{-4ix}.

$$y_h(x) = A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix}$$

Recall that we want $y_h(x)$ to be real (and remember that A & B are complex constants). To say that y_h is real is the same thing as saying that it is equal to its complex conjugate:

$$y_h(x) = \overline{y_h(x)}$$

i.e.

$$(1) \quad A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix} = \overline{A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix}}$$

$$= \overline{A e^{-3x} e^{4ix}} + \overline{B e^{-3x} e^{-4ix}} \quad (\text{since } \overline{a+b} = \bar{a} + \bar{b})$$

$$= \bar{A}\, \overline{e^{-3x}}\, \overline{e^{4ix}} + \bar{B}\, \overline{e^{-3x}}\, \overline{e^{-4ix}} \quad (\text{since } \overline{a \cdot b} = \bar{a} \cdot \bar{b})$$

$$(2) \quad = \bar{A} e^{-3x} e^{-4ix} + \bar{B} e^{-3x} e^{4ix}$$

$$(1) = (2) \iff A e^{-3x} e^{4ix} + B e^{-3x} e^{-4ix} = \bar{A} e^{-3x} e^{-4ix} + \bar{B} e^{-3x} e^{4ix}$$

in other words, the above amounts to imposing $y_h(x) = \overline{y_h(x)}$ i.e. that $y_h(x)$ is real.

$$\text{i.e. } 0 = e^{-3x} \left[A e^{4ix} + B e^{-4ix} - \bar{A} e^{-4ix} - \bar{B} e^{4ix} \right]$$

$$= e^{-3x} \left[(A - \bar{B}) e^{4ix} + (B - \bar{A}) e^{-4ix} \right]$$

Note that the above must be true for all x's. Since $e^{-3x} \neq 0$, we have

$$(A - \bar{B}) e^{4ix} + (B - \bar{A}) e^{-4ix} = 0$$

i.e. a linear combination of e^{4ix} & e^{-4ix} equal to 0. Since e^{4ix} & e^{-4ix} are linearly independent (they're not proportional to one another), we get $A = \bar{B}$ & $B = \bar{A}$ i.e. A & B are complex conjugate of one another.