Linear differential equations with constant coefficients (continued)

Example 2 : $y'' + 6y' + 25y = 0$

1/ Characteristic equation : $\lambda^2 + 6\lambda + 25 = 0$
$\iff \lambda = -3 \pm 4i$

2/ General solution:

$$y_h(x) = A\, e^{-3x} (\cos(4x) + i \sin(4x)) + B\, e^{-3x} (\cos(4x) - i \sin(4x))$$

with $A = \overline{B}$.

Set $A = \alpha + i\beta$ $\qquad B = \alpha - i\beta$

Then,
$$y_h(x) = (\alpha + i\beta)\, e^{-3x} \left(\cos(4x) + i\sin(4x)\right) + (\alpha - i\beta)\, e^{-3x} \left(\cos(4x) - i\sin(4x)\right)$$

$$= e^{-3x} \left(\alpha\cos(4x) + i\beta\cos(4x) + i\alpha\sin(4x) - \beta\sin(4x) + \alpha\cos(4x) - i\beta\cos(4x) - i\alpha\sin(4x) - \beta\sin(4x)\right)$$

$$= e^{-3x} \left(2\alpha\cos(4x) - 2\beta\sin(4x)\right)$$

$$= e^{-3x} \left(C_1\cos(4x) + C_2\sin(4x)\right)$$

Where C_1 & C_2 are arbitrary <u>real</u> constants.

$y_1(x) = e^{-3x}\cos(4x)$ & $y_2(x) = e^{-3x}\sin(4x)$ are 2 linearly independent solutions.

3/ <u>Impose initial condition</u> : $y(0)=0 \quad y'(0)=1$

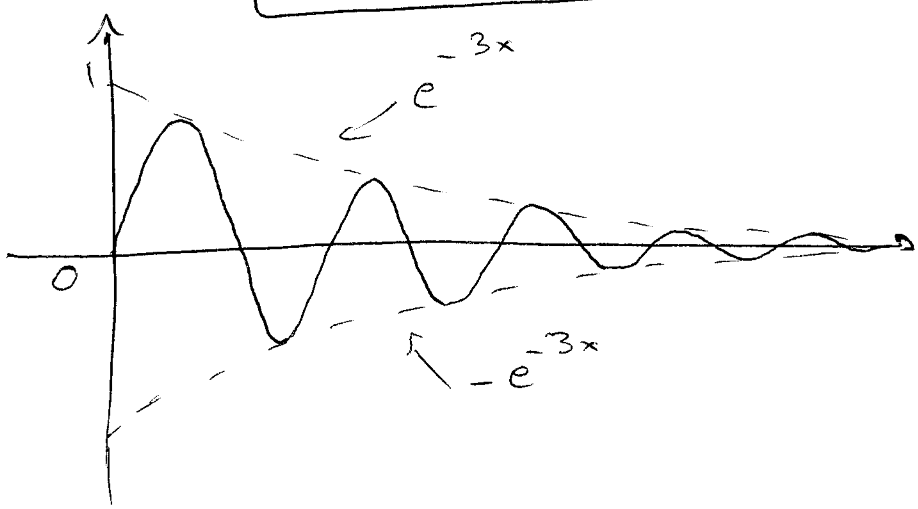
$$y(x)=C_1\, e^{-3x}\cos(4x)+C_2\, e^{-3x}\sin(4x)$$

$y(0)=C_1=0$ Then $y(x)=C_2\, e^{-3x}\sin(4x)$

$$y'(x)=-3C_2\, e^{-3x}\sin(4x)+4C_2\, e^{-3x}\cos(4x)$$

$$1=y'(0)=4C_2 \Rightarrow C_2=\frac{1}{4}$$

Therefore, $\boxed{y(x)=\frac{1}{4}\, e^{-3x}\sin(4x)}$.



<u>Example 3</u> : $y''+2y'+y=0$ with $y(0)=1;\ y'(0)=1$

1/ Characteristic equation: $\lambda^2+2\lambda+1=0$
i.e. $(\lambda+1)^2=0 \iff \lambda=-1$.
We have 1 solution, $y_1(x)=e^{-x}$.

2/ We need to find another solution, $y_2(x)$, linearly independent from $y_1(x)$.

To find $y_2(x)$, use <u>variation of parameters</u> (constants)

We know that $y(x) = C_1 y_1(x) = C_1 e^{-x}$ is a solution. We look for $y_2(x)$ in the form

$$y_2(x) = u(x) e^{-x} \quad (\text{i.e. } u(x) y_1(x))$$

Substitute this into $y'' + 2y' + y = 0$.

$$y_2'(x) = u' e^{-x} - u e^{-x} = (u' - u) e^{-x}$$

$$y_2''(x) = u'' e^{-x} - u' e^{-x} - u' e^{-x} + u e^{-x}$$
$$= (u'' - 2u' + u) e^{-x}$$

$$0 = y_2'' + 2y_2' + y_2 = (u'' - 2u' + u) e^{-x} + 2(u' - u) e^{-x} + u e^{-x}$$
$$= u'' e^{-x} \Rightarrow u'' = 0$$

So $u' = a$ and $u = ax + b$ a, b are constants.

So $y_2(x) = (ax + b) e^{-x}$. We need to choose 1 $y_2(x)$, linearly independent from $y_1(x)$.

Choose $y_2(x) = x e^{-x}$ (i.e. $a = 1$, $b = 0$).

Therefore, our 2 linearly independent solutions are
$$y_1(x) = e^{-x}; \quad y_2(x) = x e^{-x}$$

The general solution is $y(x) = C_1 e^{-x} + C_2 x e^{-x}$.

3/ Initial conditions: $y(0)=1$; $y'(0)=1$

$$y(x) = C_1 e^{-x} + C_2 x e^{-x}$$

$y(0) = 1 \Leftrightarrow 1 = C_1$

$$y'(x) = -C_1 e^{-x} + C_2 e^{-x} - C_2 x e^{-x}$$

$y'(0) = 1 \Leftrightarrow 1 = -C_1 + C_2$

$\Leftrightarrow C_2 = 1 + C_1 = 2$

Therefore, $\boxed{y(x) = e^{-x} + 2 x e^{-x}}$.

Why does this work?

$$a y'' + b y' + c y = 0$$

Assume $y_1(x)$ is a solution (that we know).

Look for another solution $y_2(x)$, in the form

$y_2(x) = u(x)\, y_1(x)$.

Substitute this into the equation:

$$y_2'(x) = y_2' = u' y_1 + u y_1'$$

$$y_2''(x) = u'' y_1 + u' y_1' + u' y_1' + u y_1''$$
$$= u'' y_1 + 2u' y_1' + u y_1''.$$

$$a y_2'' + b y_2' + c y_2 = 0$$

$$\Leftrightarrow \; a\,(u'' y_1 + 2u' y_1' + u y_1'') + b\,(u' y_1 + u y_1') + c\,u\,y_1 = 0$$

$$\Leftrightarrow \; u''(a y_1) + u'(a 2 y_1' + b y_1) + u\,(\underbrace{a y_1'' + b y_1' + c y_1}_{0}) = 0$$

$$\Leftrightarrow \; u''(a y_1) + u'(2a y_1' + b y_1) = 0$$

Let $v = u'$; then $v' = u''$

and $\quad v'(a y_1) + v\,(2a y_1' + b y_1) = 0$

1st order, linear in v $\Rightarrow$ we can always write down a solution.