Linear differential equations with constant coefficients (continued)

2. Non-homogeneous equations

2.1. Variation of parameters

$$ay'' + by' + cy = f(x)$$

Assume that you know 2 linearly independent solutions to the homogeneous equation
$$ay'' + by' + cy = 0.$$

Then, the general solution to the homogeneous equation is
$$y_h(x) = C_1 y_1(x) + C_2 y_2(x)$$

2 linearly independent solutions to $ay''+by'+cy=0$.

We are going to look for a particular solution to the non-homogeneous equation in the form
$$y_p(x) = u_1(x) y_1(x) + u_2(x) y_2(x)$$

We are going to substitute y_p into the ODE and obtain one equation for u_1 & u_2. We impose a second relationship: $u_1'(x)\, y_1(x) + u_2'(x)\, y_2(x) = 0$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$= \underbrace{(u_1' y_1 + u_2' y_2)}_{0} + u_1 y_1' + u_2 y_2'$$

$$= u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2''$$

Substitute back into ODE:

$$a y_p'' + b y_p' + c y_p = f(x)$$

i.e. $a\,(u_1' y_1' + u_1 y_1'' + u_2' y_2' + u_2 y_2'')$

$$+ b\,(u_1 y_1' + u_2 y_2') + c\,(u_1 y_1 + u_2 y_2) = f($$

i.e. $u_1 \underbrace{(a y_1'' + b y_1' + c y_1)}_{0} + u_2 \underbrace{(a y_2'' + b y_2' + c y_2}_{0}$

$$+ a\,(u_1' y_1' + u_2' y_2') = f(x)$$

i.e. $\quad u_1' y_1' + u_2' y_2' = \dfrac{1}{a} f(x)$

Therefore, we have a system of 2 first order equations for u_1 & u_2:

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = \frac{1}{a} f(x) \end{cases}$$

$$\delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

If $\delta \neq 0$, $\quad u_1' = \frac{1}{\delta} \begin{vmatrix} 0 & y_2 \\ \frac{1}{a} f(x) & y_2' \end{vmatrix} = -\frac{y_2 f}{a\delta}$

$$u_2' = \frac{1}{\delta} \begin{vmatrix} y_1 & 0 \\ y_1' & \frac{1}{a} f(x) \end{vmatrix} = \frac{y_1 f(x)}{a\delta}$$

Recall that y_1, y_2, and δ are functions of x.

<u>Can δ be 0 ?</u>

Assume $\delta \neq 0$ then let

$$c_1 y_1 + c_2 y_2 = 0$$

take derivative of this : $c_1 y_1' + c_2 y_2' = 0$

i.e. $\begin{cases} c_1 y_1 + c_2 y_2 = 0 \\ c_1 y_1' + c_2 y_2' = 0 \end{cases}$ $\qquad \delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

If $\delta \neq 0$, then we see that y_1 & y_2 are linearly independent.

The converse is not true: we can have $\delta = 0$ and y_1 & y_2 linearly independent:

<u>Ex</u> $y_1(x) = x|x|$ on $(-1,1)$
$y_2(x) = x^2$ on $(-1,1)$

On $(-1,0]$ $y_1(x) = -x^2$ $y_2(x) = x^2$
On $[0,1)$ $y_1(x) = x^2$ $y_2(x) = x^2$

So on $(-1,0]$ since y_1 & y_2 are proportional to one another, $\delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$

and on $[0,-1)$, y_1 & y_2 are the same, so δ is 0 as well.

In other words, $\delta = 0$ on $(-1,0]$ and $[0,1)$, i.e. $\delta = 0$ on $(-1,1)$.

<u>However</u> if now we add that y_1 & y_2 are solutions to a <u>linear equation</u> with continuous coefficients, then saying that y_1 & y_2 are linearly independent is the same as saying that $\delta = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$.

Because of this, since we have assumed tha y_1 & y_2 are linearly independent solutions to the homogeneous equation, we know that $\delta \neq 0$.

So our method works all the time!