Linear differential equations with constant coefficients (continued)

2. Method of undetermined coefficients

Example 1: $y'' - 4y' + 4y = 12xe^{2x}$

$\alpha = 2 \quad \beta = 0$

Is 2 a root of $\lambda^2 - 4\lambda + 4 = 0$?
Yes, it has multiplicity 2, since
$\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$

Trial solution:

$$y_p = x^2(ax+b)e^{2x} = (ax^3 + bx^2)e^{2x}$$

$$y_p' = (3ax^2 + 2bx)e^{2x} + 2(ax^3 + bx^2)e^{2x}$$
$$= (2ax^3 + (3a+2b)x^2 + 2bx)e^{2x}$$

$$y_p'' = (6ax^2 + 2(3a+2b)x + 2b)e^{2x}$$
$$+ 2(2ax^3 + (3a+2b)x^2 + 2bx)e^{2x}$$

$$= (4ax^3 + x^2(6a + 6a + 4b) + x(6a + 4b + 4b) + 2b)e^{2x}$$

$$= (4ax^3 + (12a+4b)x^2 + (6a+8b)x + 2b)e^{2x}$$

Substitute back into ode:

$$y_p'' - 4y_p' + 4y_p = 12x\,e^{2x}$$

reads

$$4ax^3 + (12a+4b)x^2 + (6a+8b)x + 2b$$
$$-4\left(2ax^3 + (3a+2b)x^2 + 2bx\right) + 4\left(ax^3 + bx^2\right)$$
$$= 12x$$

i.e. $x^3\left(\underbrace{4a - 8a + 4a}_{0}\right) + x^2\left(\underbrace{12a + 4b - 12a - 8b + 4b}_{0}\right)$
$+ x\,(6a + 8b - 8b) + 2b = 12x$

We want $\begin{cases} 6a = 12 \\ 2b = 0 \end{cases}$ i.e. $\begin{cases} a = 2 \\ b = 0 \end{cases}$

So $\boxed{y_p(x) = 2x^3 e^{2x}}$.

Example 2: $y'' + 6y' + 25y = \cos(4x)$

$\alpha = 0$ $\beta = 4$ Is $4i$ a root of $\lambda^2 + 6\lambda + 25 = 0$?

$$\lambda^2 + 6\lambda + 25 = (\lambda+3)^2 - 9 + 25 = (\lambda+3)^2 + 16$$

roots of characteristic equation: $\lambda = -3 \pm 4i$

Since $4i$ is not a root of the characteristic equation, try

$$y_p = a\cos(4x) + b\sin(4x).$$

polynomials of degree 0

$$y_p' = -4a\sin(4x) + 4b\cos(4x)$$

$$y_p'' = -16a\cos(4x) - 16b\sin(4x)$$

Substitute back into $y_p'' + 6y_p' + 25y_p = \cos(4x)$:

$$\begin{aligned}&-16a\cos(4x) - 16b\sin(4x)\\ &+ 6\left(-4a\sin(4x) + 4b\cos(4x)\right)\\ &+ 25\left(a\cos(4x) + b\sin(4x)\right) = \cos(4x)\end{aligned}$$

i.e. $\cos(4x)\left(-16a + 24b + 25a\right) + \sin(4x)\left(-16b - 24a + 25b\right.$
$= \cos(4x)$

Set $\begin{cases} 9a + 24b = 1 \\ -24a + 9b = 0 \end{cases} \Rightarrow \begin{cases} 9a + 24\frac{8}{3}a = 1 \\ b = \frac{24}{9}a = \frac{8}{3}a \end{cases}$

i.e. $\begin{cases} 9a + 64a = 1 \\ b = \frac{8}{3}a \end{cases}$ i.e. $\begin{cases} a = \frac{1}{73} \\ b = \frac{8}{3} \cdot \frac{1}{73} \end{cases}$

So $\boxed{y_p = \frac{1}{73}\left(\cos(4x) + \frac{8}{3}\sin(4x)\right)}$.