Linear differential equations with constant coefficients (continued)

Example 3: $y'' + 6y' + 25y = \cos(4x) + x$

Assume we have a particular solution y_1 to
$$y'' + 6y' + 25y = \cos(4x)$$

and a particular solution y_2 to
$$y'' + 6y' + 25y = x$$

Then
$$\begin{cases} y_1'' + 6y_1' + 25y_1 = \cos(4x) \\ y_2'' + 6y_2' + 25y_2 = x \end{cases}$$

Add the 2 equations:
$$y_1'' + y_2'' + 6y_1' + 6y_2' + 25y_1 + 25y_2 = \cos(4x) + x$$

i.e. $(y_1 + y_2)'' + 6(y_1 + y_2)' + 25(y_1 + y_2) = \cos(4x) + x$

Set $y_p = y_1 + y_2$. Then

$$y_p'' + 6y_p' + 25y_p = \cos(4x) + x$$

i.e. y_p is a particular solution to
$$y'' + 6y' + 25y = \cos(4x) + x$$

Thus, a particular solution to

$$y'' + 6y' + 25y = \cos(4x) + x$$

will be of the form

$$y_p = \underbrace{A\cos(4x) + B\sin(4x)}_{\substack{\alpha=0 \ \beta=4 \\ 4i \text{ is not a} \\ \text{root of } \lambda^2+6\lambda+25=0}} + \underbrace{Cx + D}_{\substack{\alpha=0 \ \beta=0 \\ 0 \text{ is not a} \\ \text{root of} \\ \lambda^2+6\lambda+25=0}}$$

We've in fact already found a particular solution to $y''+6y'+25y = \cos(4x)$. This was $y_{p_1} = \frac{1}{73}\left(\cos(4x) + \frac{8}{3}\sin(4x)\right)$.

We look for y_{p_2}, which is a particular solution to $y''+6y'+25y = x$, in the form

$$y_{p_2} = Cx + D.$$

$$y_{p_2}' = C \qquad y_{p_2}'' = 0$$

$$y_{p_2}'' + 6y_{p_2}' + 25y_{p_2} = x$$

$$\Leftrightarrow 0 + 6C + 25(Cx + D) = x$$

$$\Leftrightarrow 25Cx + (6C + 25D) = x$$

$$\Leftrightarrow \begin{cases} 25C = 1 \\ 6C + 25D = 0 \end{cases} \Leftrightarrow \begin{cases} C = \frac{1}{25} \\ D = \frac{1}{25}(-6C) = \frac{-6}{625} \end{cases}$$

So $y_{p_2} = \frac{1}{25}\left(x - \frac{6}{25}\right)$.

A particular solution to

$$y'' + 6y' + 25y = \cos(4x) + x$$

is $$y_p = \frac{1}{73}\left(\cos(4x) + \frac{8}{3}\sin(4x)\right) + \frac{1}{25}\left(x - \frac{6}{25}\right)$$