Other types of linear equations (continued)

I. Cauchy-Euler equations (continued)

Recall $\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dt}$; $\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$

So $x\frac{dy}{dx} = \frac{dy}{dt}$

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

This change of variable will therefore transform the Cauchy-Euler equation into an equation with constant coefficients.

Example: $t^2\frac{d^2y}{dt^2} + 7t\frac{dy}{dt} + 25y = \cos(4\ln(t)) + \ln(t)$

Let $t = e^x$, $(t>0)$

$$t\frac{dy}{dt} = \frac{dy}{dx} \quad ; \quad t^2\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} - \frac{dy}{dx}$$

So the ODE becomes:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 7\frac{dy}{dx} + 25y = \cos(4x) + x$$

i.e. $\frac{d^2y}{dt^2} + 6\frac{dy}{dx} + 25y = \cos(4x) + x$

We've solved this before. The general

solution is

$$y(x) = c_1 e^{-3x} \cos(4x) + c_2 e^{-3x} \sin(4x) + \frac{1}{73}\left(\cos(4x) + \frac{8}{3}\sin(4x)\right) + \frac{1}{25}\left(x - \frac{6}{25}\right)$$

Therefore, the general solution to the Cauchy-Euler equation is

Recall $t = e^x$ so $x = \ln(t)$

$$e^{-3x} = e^{-3\ln(t)} = t^{-3} = \frac{1}{t^3}$$

$$y(t) = \frac{c_1}{t^3} \cos(4\ln(t)) + \frac{c_2}{t^3} \sin(4\ln(t)) + \frac{1}{73}\left(\cos(4\ln(t)) + \frac{8}{3}\sin(4\ln(t))\right) + \frac{1}{25}\left(\ln(t) - \frac{6}{25}\right)$$

II Equations of order higher than 2, with constant coefficients.

Example: $y''' - 5y'' + 12y' - 8y = e^{3t}\cos(2t) + 7t\,e^t$

Characteristic equation: $\lambda^3 - 5\lambda^2 + 12\lambda - 8 = 0$

i.e. $(\lambda - 1)(\lambda^2 - 4\lambda + 8) = 0$

i.e. $(\lambda - 1)((\lambda - 2)^2 + 4) = 0$

The roots are: $\lambda = 1$ $\quad \lambda = 2 \pm 2i$