Linear systems (continued)

2. Matrix formulation (continued)

Example: $y'' - 5y' + 6y = 0$

Solve as a 2^{nd} order ODE

$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$

$$y = C_1 e^{2t} + C_2 e^{3t}$$
$$y' = 2C_1 e^{2t} + 3C_2 e^{3t}$$

$$\begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} C_1 e^{2t} + C_2 e^{3t} \\ 2C_1 e^{2t} + 3C_2 e^{3t} \end{bmatrix}$$
$$= \begin{bmatrix} C_1 e^{2t} \\ 2C_1 e^{2t} \end{bmatrix} + \begin{bmatrix} C_2 e^{3t} \\ 3C_2 e^{3t} \end{bmatrix}$$
$$= C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}$$

Solve as a system

$$y' = v$$
$$y'' = v' = 5y' - 6y = 5v - 6y$$

i.e. $$\frac{d}{dt}\begin{bmatrix} y \\ v \end{bmatrix} = \begin{bmatrix} y \\ v \end{bmatrix}'$$
$$= \begin{bmatrix} v \\ -6y + 5v \end{bmatrix}$$
$$= \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} y \\ v \end{bmatrix}}_{Y}$$

i.e. $Y' = MY$

We impose $\det(M - \lambda I) = 0$

$$M - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -6 & 5-\lambda \end{bmatrix}$$

$$\det(M - \lambda I) = -\lambda(5 - \lambda) + 6$$
$$= \lambda^2 - 5\lambda + 6$$

So $\lambda = 2$ & $\lambda = 3$ are eigenvalues of M.

Find the corresponding <u>eigenvectors</u>:

<u>$\lambda=2$</u> $\quad M-\lambda I = M-2I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$$

We want to find a vector $\xi_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ such that

$$\begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} -2\alpha+\beta=0 \\ -6\alpha+3\beta=0 \end{cases}$$

(because $\det(M-\lambda I)=0$) $\quad \iff -2\alpha+\beta=0$

$$\iff \beta = 2\alpha$$

So $\xi_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ 2\alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ where α is arbitrary

Choose $\alpha=1$ so $\xi_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $y_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$

<u>$\lambda=3$</u> $\quad M-3I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix}$

Look for $\xi_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ such that $(M-3I)\xi = 0$

i.e. $\begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{cases} -3\alpha+\beta=0 \\ -6\alpha+2\beta=0 \end{cases}$

$$\iff \beta = 3\alpha$$

So $\xi_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ 3\alpha \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $y_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}$

The general solution to $Y' = MY$ is

$$Y = C_1 Y_1(t) + C_2 Y_2(t)$$

linear combination of 2 linearly independ solutions

$$= C_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}$$