Linear systems (continued)

3. Classification of the origin as a fixed point

We are interested in systems of the form

$$Y' = MY$$

when $Y = \begin{bmatrix} x \\ y \end{bmatrix}$ and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

2x2 with constant coefficients

Eigenvalues of M solve the characteristic equation

$$\lambda^2 - \lambda \, Tr(M) + \det(M) = 0$$

Recall that if λ_1 & λ_2 are roots of the above equation, then

$$\lambda^2 - \lambda \, Tr(M) + \det(M) = (\lambda - \lambda_1)(\lambda - \lambda_2)$$
$$= \lambda^2 - \lambda(\lambda_1 + \lambda_2) + \lambda_1 \lambda_2$$

So $Tr(M) = \lambda_1 + \lambda_2$
$\det(M) = \lambda_1 \lambda_2$

We want to describe the trajectories of $Y' = MY$ in the vicinity of the origin, which is a fixed point. (Origin is the same as $Y = 0$).

1. <u>Real roots</u> $Tr(M)^2 - 4\det(M) > 0$

* Roots of same sign : $\det(M) > 0$
Both roots negative : $Tr(M) < 0 \rightarrow$ sink
" positive : $Tr(M) > 0 \rightarrow$ source

* Roots of opposite signs : $\det(M) < 0$

2. <u>Complex roots</u> $Tr(M)^2 - 4\det(M) < 0$, $\det(M) > 0$
Recall the roots are complex conjugate of one another since M has real entries.

* $\lambda = \alpha \pm i\beta$ $\alpha \neq 0$
$\alpha > 0$ unstable spiral $Tr(M) > 0$
$\alpha < 0$ stable spiral $Tr(M) < 0$
* $\lambda = \pm i\beta$ center $Tr(M) = 0$

3. <u>Degenerate cases</u>
<u>Double (real) root</u>. $\lambda_1 = \lambda_2 = \lambda$ $Tr(M)^2 = 4\det(M)$
We need Y_1 & Y_2 as 2 linearly independent solutions to $Y' = MY$.
In this case, since $\lambda_1 = \lambda_2$, we need to look at the eigenvectors ξ_1 & ξ_2.
Recall $\lambda\xi = M\xi$.
- 2 linearly independent eigenvectors ξ_1 & ξ_2, then we have a star.
$Tr(M) < 0$ stable star
$Tr(M) > 0$ unstable star

- What if one of the roots is 0 :

  $\det(M) = 0 \qquad \lambda_1 = 0 \; ; \; \lambda_2 = \mathrm{Tr}(M) \in \mathbb{R}$

$$Y(t) = C_1 e^{\lambda_1 t} \zeta_1 + C_2 e^{\lambda_2 t} \zeta_2$$

$$= C_1 \zeta_1 + C_2 e^{\lambda_2 t} \zeta_2$$

Choose $C_2 = 0$: we have a line of fixed points !