Calculus and Differential Equations II

MATH 250 B

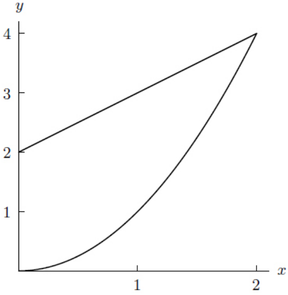
Area, volume, arc length, density, and center of mass

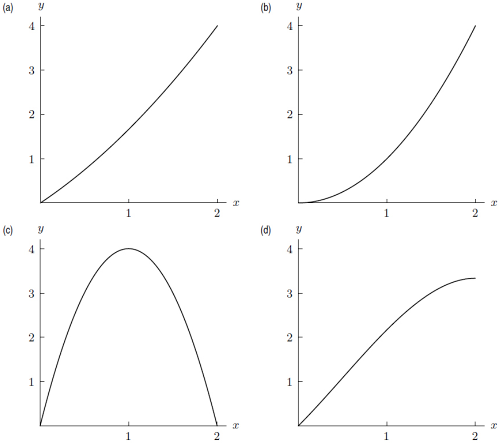
Areas and volumes

- Example 1: We want to calculate the area of the region between the curves of equation $y = x$ and $y = \sqrt{x}$, for $x \in [0, 1]$.
- Draw a picture and calculate the area using
  1. Horizontal slices
  2. Vertical slices
- Select your answer
  1. $1/2$
  2. $1/3$
  3. $1/6$
  4. $1/8$
- Example 2: Find the volume of a prism with sides of length $2, 3$ and 4 centimeters.
- Recall that if you have to choose how to slice an object, try to make your task as simple as possible.

Areas and volumes (continued)

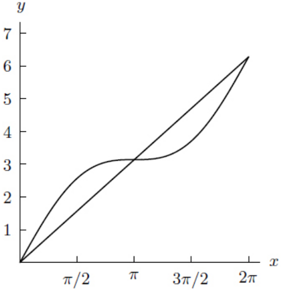
Which of the graphs below represent the area, as a function of x, of the region between the two curves shown in the plot on the left?

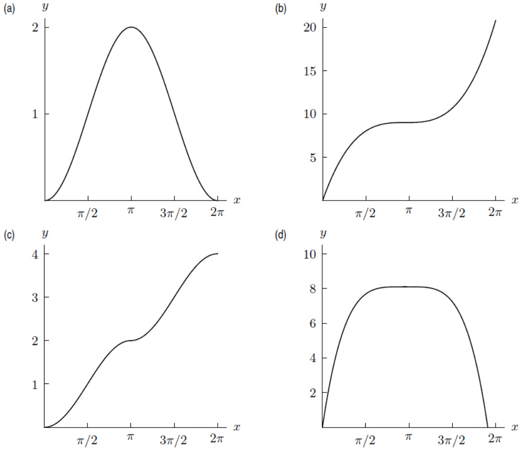




Areas and volumes (continued)

Which of the graphs below represent the area, as a function of x, of the region between the two curves shown in the plot on the left?

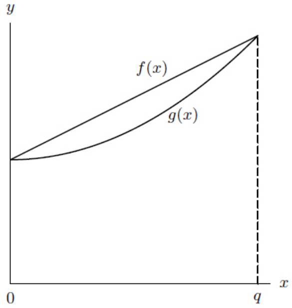




Volumes

- Example 1: Find the volume of the object obtained by rotating the region bounded by $y = \sqrt{x}$, $x = 1$, and $y = 0$, about the axis of equation $x = 1$.
- Example 2: Find the volume of the object obtained by rotating the region $\mathcal{R}$ bounded by $y = \exp(x)$, $x = 0$, $x = 1$, and the x-axis, about the line of equation $y = 7$.
- Example 3: Find the volume of the object whose base is the region $\mathcal{R}$ defined above, and whose cross-sections perpendicular to the x-axis are squares.

Which formula represents the volume of the solid obtained by rotating the region between the 2 curves about the y-axis?



1. $\int_0^q 2\pi x\,(f(x) - g(x))\;dx$
2. $\int_0^q (f(x) - g(x))\;dx$
3. $\int_0^q \pi\,(f(x) - g(x))^2\;dx$
4. $\int_0^q \left(\pi f(x)^2 - \pi g(x)^2\right)\;dx$
5. $\int_0^q \pi x\,(f(x) - g(x))\;dx$

Arc length

- To compute the length of a curve, think of a particle moving along the curve and integrate its velocity as a function of time.
- Of course, the above assumes that the particle always moves in the same direction along the curve, i.e. that its speed does not change sign.
- Then, if the curve is given as the graph of a function, say $y = f(x)$ for $x \in [a, b]$, then its length l is

$$l = \int_a^b \sqrt{1 + f'(x)^2}\, dx.$$

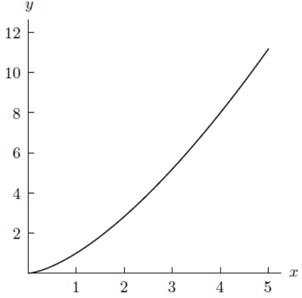
- Alternatively, if the curve is given in parametric form, i.e. if we know $x(t)$ and $y(t)$, for $t \in [t_1, t_2]$, then

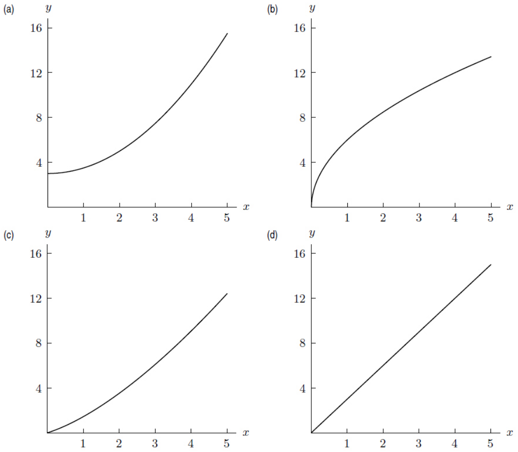
$$l = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

- Example 1: Consider a curve described by $x = t$ and $y = f(t)$. Do the two formulas given above match?
- Example 2: Find the arc length of the curve of equation $y = x^{3/2}$, for $x \in [0, 2]$, and indicate which of the possible answers below is correct.
  1. $\frac{8}{27}\left(\left(\frac{11}{2}\right)^{3/2} - 1\right)$
  2. $\frac{10}{27}\left(\left(\frac{11}{2}\right)^{5/2} - 1\right)$
  3. $\frac{8}{27}\left(\left(\frac{11}{2}\right)^{5/2} - 1\right)$

Arc length (continued)

Which of the graphs below represents the arc length of the curve shown on the left?





Density and center of mass

- Here, we only consider objects that are one-dimensional. As a consequence, their density is given in units of mass per unit length (e.g. g/cm).
- Example: Find the mass of a rod of length 10 cm, and of density $\delta(x) = exp(-x)$ grams per centimeter, for $x \in [0, 10]$.
- The center of mass of a collection of N particles of mass m_i, at positions x_i is the point with coordinate

$$\bar{x} = \frac{\sum_{i=1}^{N} x_i m_i}{\sum_{i=1}^{N} m_i}.$$

- By analogy, the center of mass of a rod of density $\delta(x)$, starting at $x = a$ and ending at $x = b$ has coordinate

$$\bar{x} = \frac{\int_a^b x\delta(x)\ dx}{\int_a^b \delta(x)\ dx}.$$