Calculus and Differential Equations II

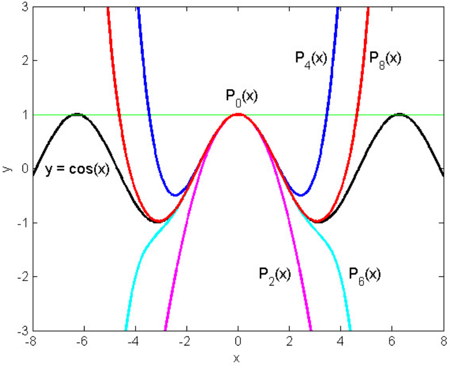
MATH 250 B

Series expansions and linear combinations

Taylor polynomials

Recall that the Taylor polynomial of degree n, centered at $x = a$, of a function f is given by

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a).$$



The figure on the left shows the graphs of $\cos(x)$ and of its Taylor polynomials of degree up to 8, near $x = 0$.

Link to d'Arbeloff Taylor Polynomials software

Taylor polynomials (continued)

$$P_n(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a).$$

- Recall that the error R_n made by replacing f by its Taylor polynomial P_n is such that

$$f(x) = P_n(x) + R_n(x), \quad R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}f^{(n+1)}(\xi), \quad \xi \in (a,x).$$

- A Taylor polynomial may be viewed as the partial sum of a power series. If the function f is infinitely differentiable, it is natural to ask what happens to P_n as $n \to \infty$.
- We define the Taylor series of a smooth function f near $x = a$ as

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor series

- Since a Taylor series is a power series, it has an interval of convergence.
- Example 1: Find the Taylor series of $\arctan(x)$ near $x = 0$ and find its radius of convergence.
- Example 2: The binomial series is the Taylor series for $(x+1)^p$ near $x = 0$. Find its radius of convergence.
- Taylor series may be found by substituting a Taylor series into another one.
- Example 3: Find the Taylor series of $\cos(x^2)$ and of $e^x \cos(x)$.
- Taylor series may also be found by differentiation and by integration.
- Example 4: Find the Taylor series of $\arctan(x)$ using term-by-term integration.

Expansions of functions onto other functions

- Taylor series provide an example of how a smooth function can be expanded onto a family of functions, in this case polynomials.
- The resulting infinite expansion is only valid in the interval of convergence of the series.
- Another way of looking at this is as follows: given a smooth function f, one may ask whether it is possible to write f as an infinite linear combination of functions in the set $\{1,\ x,\ x^2,\ x^3,\ \dots\ x^n,\ \dots\}$.
- The Taylor series of f near $x = 0$, together with the associated interval of convergence, provide an answer to this question.

Expansions of functions onto other functions (continued)

- The one-dimensional wave equation models the 2-dimensional dynamics of a vibrating string which is stretched and clamped at its end points (say at $x = 0$ and $x = L$).

- The function $u(x,t)$ measures the deflection of the string and satisfies

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c^2 \propto T, \;\; T \equiv \text{tension of the string}$$

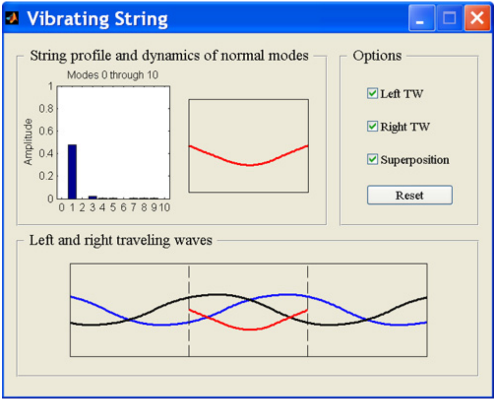
  with Dirichlet boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad \text{for all } t \geq 0.$$

- In what follows, we assume that the initial conditions are

$$u(x,0) = f(x), \quad u_t(x,0) \equiv \frac{\partial u}{\partial t}(x,0) = g(x), \quad \text{for } x \in [0,\; L].$$

Expansions of functions onto other functions (continued)



- The figure on the left shows a MATLAB GUI that describes the dynamics of an elastic string in the absence of damping.
- The simulation reproduces many features observed in experiments.

Expansions of functions onto other functions (continued)

- Solving the wave equation with the above initial and boundary conditions gives solutions of the form

$$u(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin\left(n\frac{\pi x}{L}\right),$$

  where the coefficients $C_n(t)$ are functions of time that can be calculated.

- In other words, the solution to the wave equation can be written as an infinite sum of sine functions. This is a form of Fourier series expansion.

- You will learn how to solve the wave equation if you take a course on partial differential equations, such as MATH 322, MATH 422, or MATH 456.

Linear combinations

- A linear combination of the n functions $f_1,\ f_2,\ \cdots,\ f_n$ is an expression of the form

$$c_1 f_1 + c_2 f_2 + \cdots + c_n f_n,$$

where the c_i's are scalars.

- A Taylor series near $x = 0$ may thus be viewed as an infinite linear combination of the functions in $\{1,\ x,\ x^2,\ x^3,\ \ldots\ x^n,\ \ldots\}$.

- A set of functions $\{f_1, f_2, \cdots, f_n\}$ is linearly independent if the only linear combination equal to zero is such that all of the coefficients are equal to zero. In other words, $\{f_1, f_2, \cdots, f_n\}$ is linearly independent if and only if

$$c_1\, f_1(x) + c_2\, f_2(x) + \cdots + c_n\, f_n(x) = 0 \quad \text{for all } x$$
$$\Longrightarrow c_1 = c_2 = \cdots = c_n = 0.$$

Linear independence

- Example 1: Are x and x^2 linearly independent functions?
- Example 2: Are $\cos(x)$ and $\sin(x)$ linearly independent functions?
  1. Yes
  2. No
- Example 3: Are e^x and e^{-x} linearly independent functions?
  1. Yes
  2. No
- Example 4: Is the set $\{1, x, x^2, (1+x)^2\}$ linearly independent?
  1. Yes
  2. No

Taylor series and differential equations

- For an equation of the form $y' = g(x)$, if we cannot find an antiderivative of g, we may still get its Taylor series expansion near say $x = 0$ by integrating the Taylor series of g term by term.

- Example: Find the Taylor series of the error function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt,$$

  near $x = 0$.

- We can also find solutions to an equation of the form $y' = g(x, y)$ by substituting a series for y and looking for a recursion relation between the coefficients of the series.

- Example: Solve $y' = 2x - y$ with initial condition $y(0) = 1$.

Deformations of an elastic string

Slow motion video downloaded from the *Waves on a String* page at the University of Salford.

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