Calculus and Differential Equations II

MATH 250 B

Improper integrals

Improper integrals

- There are two types of improper integrals, those for which one of the limits of integration is infinite, and those whose integrand diverges at some point in the interval of integration.
- The improper integral $\int_a^\infty f(x)\ dx$, where f is continuous on $[a,\infty)$, is defined as

$$\int_a^\infty f(x)\ dx = \lim_{b\to+\infty} \int_a^b f(x)\ dx.$$

- If the limit exits and is finite, then one says that the integral converges. If not, one says that the integral diverges.
- Remarks:
  1. The definition of $\int_{-\infty}^c f(x)\ dx$ is similar.
  2. If $\lim_{x\to+\infty} f(x) = a,\ a \neq 0$, then $\int_a^\infty f(x)\ dx$ cannot converge.

Improper integrals (continued)

- Example 1: Does $\int_1^\infty e^x \, dx$ converge?
  1. Yes
  2. No

- Example 2: Does $\int_1^\infty \frac{1}{x^3} \, dx$ converge?
  1. Yes
  2. No

- Example 3: When does $\int_1^\infty \frac{1}{x^p} \, dx$ converge?

- Answer:
  - If $p > 1$, then $\int_1^\infty \frac{1}{x^p} \, dx$ converges and is equal to $\frac{1}{p-1}$.
  - If $p \leq 1$, then $\int_1^\infty \frac{1}{x^p} \, dx$ diverges.

Improper integrals (continued)

- We define $\int_{-\infty}^{+\infty} f(x)\,dx$, where f is continuous on $\mathbb{R}$, as

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{+\infty} f(x)\,dx, \qquad a \in \mathbb{R}.$$

- As a consequence, for $\int_{-\infty}^{+\infty} f(x)\,dx$ to converge, we need convergence of both $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{+\infty} f(x)\,dx$.

- Example: Can you find a function $f(x)$ such that $\int_{-\infty}^{0} f(x)\,dx$ converges but $\int_{0}^{+\infty} f(x)\,dx$ diverges?

Integrals whose integrand diverges

- Assume that the function f is continuous except at $x = a$, $a \in \mathbb{R}$, where it diverges.
- Then, integrals of the form $\int_a^c f(x)\ dx$, $\int_b^a f(x)\ dx$, or $\int_b^c f(x)\ dx$ with $a \in (b, c)$, are all improper integrals.
- As before, we have the following definition

$$\int_a^c f(x)\ dx = \lim_{z \to a^+} \int_z^c f(x)\ dx,$$

and similarly for $\int_b^a f(x)\ dx$.

- If the above limit exits and is finite, then the corresponding integral converges. If not, it diverges.

Integrals whose integrand diverges (continued)

- Example 1: Does $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ converge? If so, find its value.
- For $a \in [b, c]$ we define

$$\int_b^c f(x)\ dx = \int_b^a f(x)\ dx + \int_a^c f(x)\ dx,$$

and both integrals have to converge for $\int_b^c f(x)\ dx$ to converge.

- Example 2: Does $\int_{-1}^1 \frac{dx}{x^2}$ converge?
  1. Yes
  2. No

  If the answer is yes, find its value.

Integrals whose integrand diverges (continued)

- Example 3: For which values of p does $\int_0^1 \frac{dx}{x^p}$ converge?
- Answer:
  - If $p \geq 1$, then $\int_0^1 \frac{dx}{x^p}$ diverges.
  - If $p < 1$, then $\int_0^1 \frac{dx}{x^p}$ converges, and is equal to $\frac{1}{1-p}$.
- Example 4: Does $\int_{-1}^1 \frac{du}{u}$ converge?
  1. Yes
  2. No
- Example 5: Without integrating, can you decide whether $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$ converge?

More on the notion of convergence

- Does $\int_0^\infty \frac{x}{e^x}\,dx$ converge?
  1. Yes
  2. No

- Use your calculator and Simpson's rule to approximate this integral. In other words, evaluate $\int_a^b x\,e^{-x}\,dx$, for $b = 10,\ 15,\ 20,\ 30,\ 40.$ What do you observe?

- Similarly, for $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$, use your calculator to evaluate $\int_0^b \frac{dx}{\sqrt{4-x^2}}$, for $b = 1.9,\ 1.99,\ 1.999.$ What do you observe?

Comparison of improper integrals

- The methods discussed below are useful when you want to know whether and improper integral converges rather than its particular exact value.
- Also, if you know in advance that an integral diverges, then there is no need to try to evaluate it.
- Comparison theorems
  - If $0 \leq f(x) \leq g(x)$ and $\int_a^\infty g(x)\ dx$ converges, so does $\int_a^\infty f(x)\ dx$.
  - If $0 \leq g(x) \leq f(x)$ and $\int_a^\infty g(x)\ dx$ diverges, so does $\int_a^\infty f(x)\ dx$.

Comparison of improper integrals (continued)

- Example 1: Does $\int_1^\infty \frac{1}{1+x^3}\, dx$ converge?
  1. Yes
  2. No

- Example 2: Does $\int_4^\infty \frac{3+\sin(\alpha)}{\alpha}\, d\alpha$ converge?
  1. Yes
  2. No

- Example 3: Does $\int_1^\infty \frac{x^2+4}{x^4+3x^2+11}\, dx$ converge?
  1. Yes
  2. No