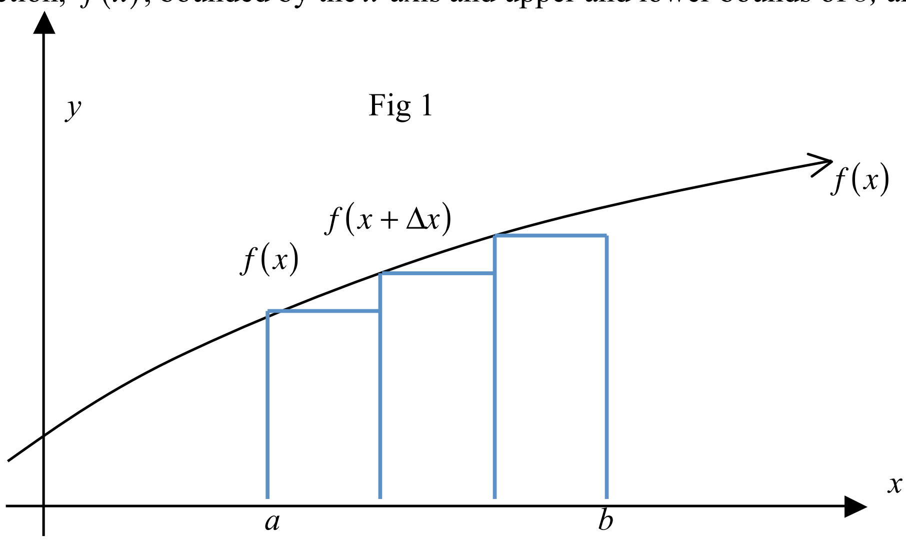
Math 250B - Section II
Tuesday, January 27, 2009

Section Focus:

1. Know and understand where the concept of the integral comes from.
2. Know how other methods of approximation come from and their accuracy.

The integral:
Geometry allows us to find the area of ideal figures, but in reality these figures don't exist. Though geometry does not directly apply to problems outside the idealized geometric world, it allows us to derive a mathematical framework to tackle such problems. The mathematical tool mentioned is the *integral*, and it is simply a sum of smaller areas. The following, Fig 1, is the graph of a function, $f(x)$, bounded by the x-axis and upper and lower bounds of b, and a.



To better establish what a and b are note that they are different x values. They determine where to start and stop adding area. If we add up n intervals or n rectangles, then we can express this sum as:

$$A \approx \sum_{i=1}^{n} f(x_i)\Delta x \quad \text{Where } \Delta x = \frac{b-a}{n}.$$

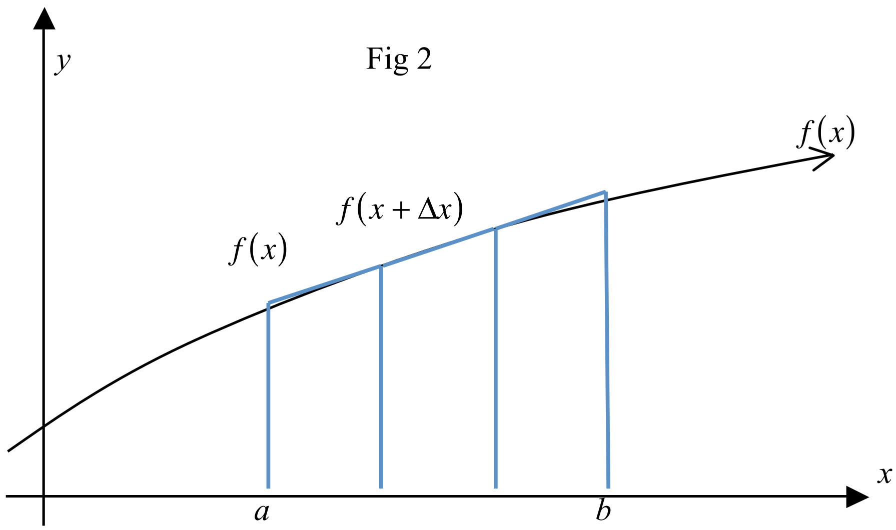
If we allow the size of the intervals to go to infinity, then n will go to infinity. This just means that if we add up an infinite number of infinitely small triangles, then we get the exact area. This is almost true.

It is important to note that the number or subintervals going to infinity does not imply that the area goes to the exact value of the area under the curve. For example, if we divide the interval into regions of different size then theoretically we could divide the initial interval in half and let the number of intervals on the left or right go to infinity. Thus it is fundamental to say that the size of the greatest interval goes to zero because it forces all smaller intervals to go to zero. Thus our mathematical expression becomes:

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(x_i)\Delta x_i = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \int_a^b f(x)dx.$$

Where $\|\Delta\|$ is the greatest sub interval, and Δx is the same as in the first expression.

The Integral and Trapezoids:
At this point it is important to ask why other figures are not used in defining the integral. The answer to this question is simply that rectangles are among the simplest geometric figures and their areas are easily attained. But if we wish to apply a more rigorous mathematical inquiry then lets choose another geometric figure to divide the interval into. Let the interval be divided into trapezoids. This situation, Fig 2, is shown below



Using the geometric definition of a trapezoid we can determine the area of the first trapezoidal region as follows:

$$\Delta A = \frac{1}{2}\Delta x\big(f(x) + f(x+\Delta x)\big).$$

At this point the same procedure of adding up all the areas as was preformed with the rectangles, but there is a small difference. If we take the Taylor expansion of $f(x+\Delta x)$, then it is seen that, after algebraic manipulation, the expression for area becomes:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{f(x_i) + f'(x_i)\Delta x + f(x_i) + \Psi(\Delta x)^2}{2}\Delta x\right].$$

Where $\Psi(\Delta x)^2$ represents the correction factor in the Taylor expansion of $f(x+\Delta x)$.

At this point, notice the presence of identical terms in the numerator. If they are grouped they can be brought out in front to produce a different representation of the area. Note that his area is the same as the first and contains terms that are being amplified by delta-*x* terms.

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f(x_i)\Delta x + \frac{f'(x_i)\Delta x + \Psi(\Delta x)^2}{2}\Delta x\right]$$

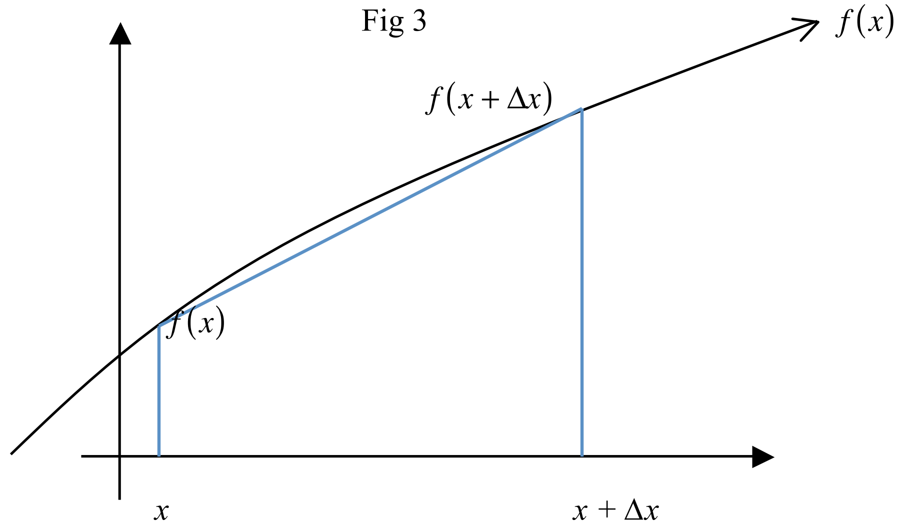
This expression can be split into two different parts using the properties of a sum. This produces:

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\left[f\left(x_i\right)\Delta x\right] + \lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{f'\left(x_i\right)\Delta x + \Psi\left(\Delta x\right)^2}{2}\Delta x\right].$$

The term to the left is easily recognized as the integral over the interval described by taking small rectangles. The term to the left, however, represents something different and goes to zero as *n* becomes infinitely large. Thus the method of trapezoids works, and reduces to the method of rectangles.

The Integral and length:
In the previous example we investigated the idea that the area defining the region under a curve can actually be represented as various shapes and when the integral or sum is computed they all become the same expression. But what about length, is it okay to say the length at some point on the curve is just delta-*x*? Lets take a look:

Fig 3



The length of the line connecting the two points is approximately that of the curve. This approximation gets better as delta-*x* goes to zero. If we use the Pythagorean expression to relate the length of the line, *s*, to that of the change in *y*, and change in *x*, then we obtain the following:

$$\Delta s^2 = \Delta y^2 + \Delta x^2$$

If we use the function and obtain a value for delta-*y*, then we see that this expression becomes very similar to the previous trapezoid example because the term $f\left(x+\Delta x\right)$ is present. This new expression is:

$$\Delta s = \sqrt{1 + f'\left(x\right) + \psi\left(\Delta x\right)}\Delta x$$

It is apparent that when we send delta-x to zero the expression becomes the classical integral expression:

$$s = \int_a^b \sqrt{1 + f'\left(x\right)}dx\,.$$