Fibonacci Sequences

A mathematician from the early 13th century, Fibonacci of Pisa, derived a mathematical sequence which follows:

$$S_n = S_{n-1} + S_{n-2} \dots$$

The so called Fibonacci sequence and corresponding Fibonacci numbers go on infinitely. The numbers, starting with n=0 follow the form:

0,1,1,2,3,5,8,13,21,34,55,89…

The Fibonacci sequence is not merely a mathematical expression, but also is used quite often in nature. Phyllotaxis is used to describe how leaves are distributed along a stem. The leaves are often distributed in an organization that is similar to the Fibonacci sequence. For some plants, flowers, and trees, the leaves are arranged in a certain fraction of a revolution around the stem. Some examples are 1/2 of a revolution, 1/3, 2/5, 3/8, 5/13, 8/13, and numerous others. These are known as phyllotactic ratios and their numerators and denominators are numbers of the Fibonacci sequence, though not always successive ones.

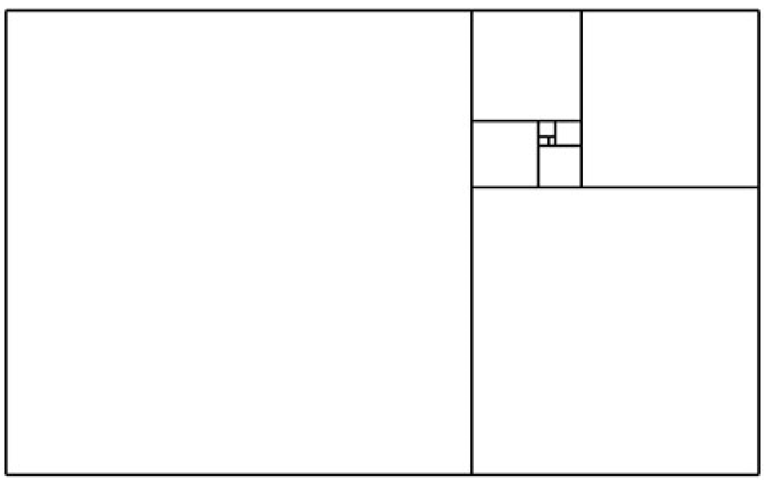
More examples of the Fibonacci sequence used in nature are the patterns of seeds on the head of a sunflower, patterns in daisies, and the spirals in pine cones. The spirals in the seeds of a sunflower have an angle of 137.5°, which is called the divergence angle and is very much related to the golden ratio. If you take the ratio of two consecutive numbers in the Fibonacci sequence, multiply that by 360°, and then subtract that from 360°, you get angles that get closer and closer to 137.5° as you get higher in the Fibonacci sequence and the ratios become more accurate. This golden angle is the only angle at which the seeds can pack together without creating any gaps. The Fibonacci sequence is present in many other aspects of sunflowers as well as nature in general. This goes to show that mathematics is everywhere in nature and not just confined to textbooks and classrooms.

Img. 1

The Golden Ratio

The Fibonacci sequence has more uses than simply a collection of numbers following the form $S_n = S_{n-1} + S_{n-2}$ ... The sequence will converge to a certain ratio of $\frac{1+\sqrt{5}}{2}$. This ratio is called the Golden Ratio, signified by $\tau = \frac{1+\sqrt{5}}{2} = 1.61803398\ldots$ This ratio was considered by the ancient Greeks to be the basis for the most beautiful things in the world. For instance, people whose face proportions conform to the golden ratio are said to be regarded as beautiful by most people. Also, many Greek buildings such as the Athens Acropolis were built with the golden ratio in mind for the dimensions.

One representation of the Golden ratio is a rectangular box that is successively split and divided according to the ratio (see **Img. 2**). The proof for derivation of the golden ratio is provided at the end of this document.



Img. 2

Fibonacci's Rabbit Problem

Fibonacci created a problem dealing with rabbit reproduction as a means for representing the Fibonacci sequence. In this problem, one initial pair of rabbits create two more rabbits during the first season. During the second season, the two pairs of rabbits create two more pairs of rabbits, but then the original pair dies off before it can reproduce for the third season. All of the rabbits reproduce for two seasons then die immediately. This goes on for a total of six seasons. A diagram of this rabbit reproduction is represented in **Chart 1**.

Diagram of Rabbits:

*Note: Each circle represents individual rabbits, not pairs of rabbits.

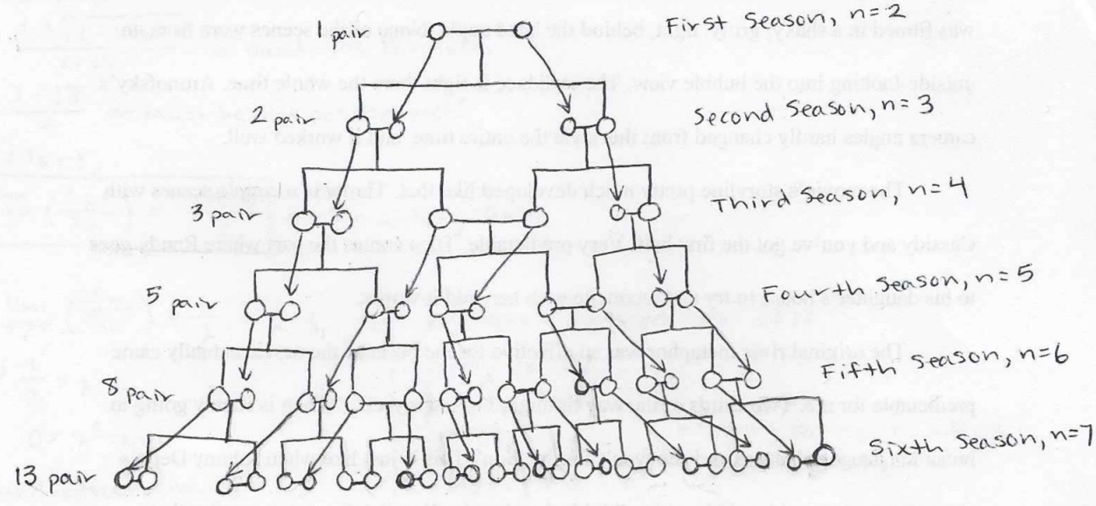


Chart 1

Proving the Golden Ratio

Sequence $\mathcal{U}_n = \frac{S_{n+1}}{S_n}$ where S_n is the Fibonacci sequence.

$$\mathcal{U}_n = \frac{3}{2} + \frac{5}{3} + \frac{8}{5} + \frac{13}{8} + \frac{21}{13} + \frac{34}{21} + \frac{55}{34} + \dots$$

$$\mathcal{U}_n = 1.5 + 1.6 + 1.6 + 1.625 + 1.615 + 1.619 + 1.618 + \dots$$

So...

$$\mathcal{U}_n = \frac{S_{n+1}}{S_n} = \frac{S_n + S_{n-1}}{S_n} = 1 + \frac{S_{n-1}}{S_n}$$

$$\underbrace{\lim_{n\to\infty} \frac{S_{n+1}}{S_n}}_{\text{Call this "x"}} = \lim_{n\to\infty}\left(1 + \frac{S_{n-1}}{S_n}\right)$$

$$\Rightarrow \quad x = \lim_{n\to\infty}\left(1 + \frac{S_{n-1}}{S_n}\right)$$

$$x = \lim_{n\to\infty}(1) + \lim_{n\to\infty}\left(\frac{S_{n-1}}{S_n}\right)$$

$$x = 1 + \lim_{n\to\infty}\left(\frac{S_{n-1}}{S_n}\right)$$

$$x - 1 = \lim_{n\to\infty}\left(\frac{S_{n-1}}{S_n}\right)$$

If $\frac{S_{n+1}}{S_n} = x$, or $\frac{\text{next population}}{\text{current population}} = x$

Then $\frac{S_{n-1}}{S_n} = \frac{1}{x}$, or $\frac{\text{previous population}}{\text{current population}} = \frac{1}{x}$

Thus...

$$x - 1 = \frac{1}{x}$$

$$x^2 - x = 1$$

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2(1)}$$ ← Quadratic Formula

$$x = \frac{1 \pm \sqrt{5}}{2}$$ ← Must be the positive root!

Therefore,

$$\boxed{\lim_{n \to \infty} \left(\frac{S_{n+1}}{S_n}\right) = \frac{1+\sqrt{5}}{2}}$$ ← The Golden Ratio

Using the general form of the Fibonacci sequence, $S_n = S_{n-1} + S_{n-2}$ ... a ratio of two successive terms creates the golden ratio, according to $\mu_n = \frac{S_{n+1}}{S_n}$. As n increases, the ratio of consecutive terms goes closer and closer to $\frac{1+\sqrt{5}}{2}$, as shown in **Chart 2**.

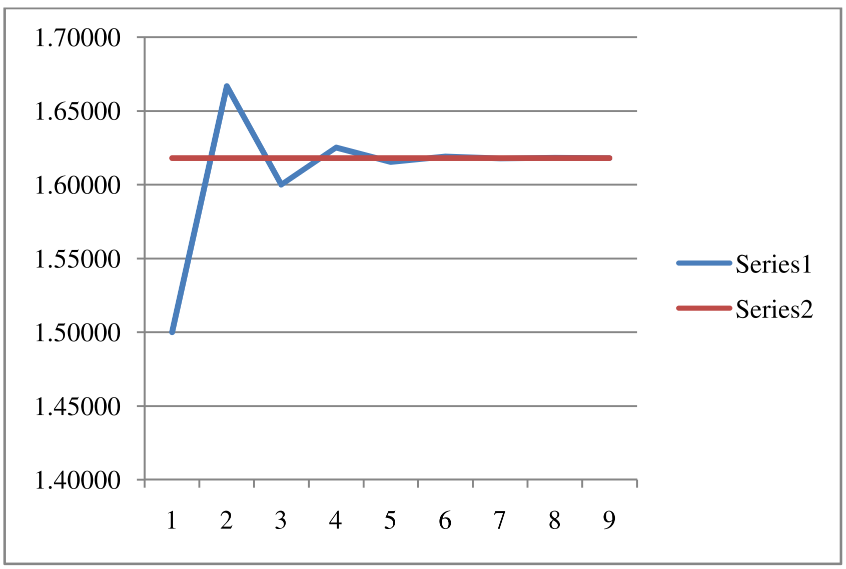


Chart 2

The remainder of the problem dealt with different aspects of the golden ratio. The proof of parts c-f can be found below. See Problem packet, question no. 2 for specifics of each question.

(c) $\lim_{n\to\infty}\left(\frac{s_n}{s_{n+1}}\right) = \frac{1}{\lambda} = \lambda - 1$

$\Rightarrow \frac{1}{\lambda} = \lambda - 1$

$0 = \lambda^2 - \lambda - 1$

Quadratic Formula:

$\lambda = \frac{1 \pm \sqrt{1+4}}{2}$

$\frac{1-\sqrt{5}}{2} < 0$

$\frac{1+\sqrt{5}}{2} > 0$

(d) So,

$\boxed{\varphi = \frac{1+\sqrt{5}}{2}}$

$\lambda = \frac{1-\sqrt{5}+\sqrt{5}-\sqrt{5}}{2}$

$\lambda = \frac{1+\sqrt{5}-2\sqrt{5}}{2}$

$\lambda = \frac{1+\sqrt{5}}{2} - \frac{\cancel{2}\sqrt{5}}{\cancel{2}}$

$\lambda = \frac{1+\sqrt{5}}{2} - \sqrt{5}$

$\lambda = \varphi - \sqrt{5}$

Another method for (d):

$x^2 - x - 1 = 0$

r_1, r_2

$r_1 \cdot r_2 = -1$

$r_1 + r_2 = 1$

* Solving for r_2, r_1 using coefficients.

$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2$

$r_1 = \varphi \quad r_2 = 1 - \varphi = -\frac{1}{6}$

φ = the number λ will converge to.

(e) $S_{n+1} = S_{n-1} + S_n$

$$\Rightarrow u_{n+1} = u_n + u_{n-1}$$

$$\alpha \varphi^{n+1} + \beta(1-\varphi)^{n+1} = \alpha\varphi^n + \beta(1-\varphi)^n + \alpha\varphi^{n-1} + \beta(1-\varphi)^{n-1}$$

$$= \alpha(\varphi^n + \varphi^{n-1}) + \beta\left[(1-\varphi)^n + (1-\varphi)^{n-1}\right]$$

*Aside: $\varphi^n + \varphi^{n-1}$

• Factor out a φ^{n+1}:

$$\varphi^n + \varphi^{n-1} = \varphi^{n+1}\left[\varphi^{-1} + \varphi^{-2}\right]$$

$$= \varphi^{n+1}\left(\frac{1}{\varphi} + \frac{1}{\varphi^2}\right)$$

$$= \varphi^{n+1}\left(\underbrace{\frac{\varphi+1}{\varphi^2}}\right)$$

*this must equal 1 if the right side will equal the left side.

So...

$$\frac{\varphi+1}{\varphi^2} = 1$$

$$\varphi^2 = \varphi + 1$$

$$* \ \varphi^2 - \varphi - 1 = 0 \ *$$

And, $\varphi^n + \varphi^{n-1} = \varphi^{n+1}$

So,

$$\alpha \varphi^{n+1} + \beta(1-\varphi)^{n+1} = \alpha\varphi^n + \beta(1-\varphi)^n + \alpha\varphi^{n-1} + \beta(1-\varphi)^{n-1}$$

$$= \alpha\underbrace{(\varphi^n + \varphi^{n-1})}_{=\varphi^{n+1}} + \beta\underbrace{\left[(1-\varphi)^n + (1-\varphi)^{n-1}\right]}_{=(1-\varphi)^{n+1}}$$

Therefore,

$$\boxed{\alpha\varphi^{n+1} + \beta(1-\varphi)^{n+1} = \alpha\varphi^{n+1} + \beta(1-\varphi)^{n+1}}$$

f) $0 = \alpha\varphi + \beta(1-\varphi) \qquad n=1$

$1 = \alpha\varphi^2 + \beta(1-\varphi)^2 \qquad n=2$

$$\left[\begin{array}{l} \alpha\varphi \pm \beta(1-\varphi)^2 \\ 1 = \varphi^2\alpha + \beta(1-\varphi)^2 \end{array}\right.$$

$$\alpha = \frac{\beta(1-\varphi)}{\varphi}$$

$$1 = -\beta(1-\varphi)(\varphi) + \beta(1-\varphi)^2$$

$$1 = -\beta 4(1-\varphi) + \beta(1-\varphi)^2$$

$$\beta = \frac{1}{(1-\varphi)^2 - \varphi(1-\varphi)}$$

$$\beta = \frac{1}{\left(\frac{1-\sqrt{5}}{2}\right)^2 - \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right)}$$

which translates to...

$$\cancel{\frac{1}{-\sqrt{5}+5}} \qquad \frac{1}{\frac{-\sqrt{5}+5}{2}} = \beta = \frac{2}{5-\sqrt{5}}$$

$$0 = \alpha\varphi + \left(\frac{2}{5\sqrt{5}}\right)(1-\varphi)$$

$$= \alpha\left(\frac{1+\sqrt{5}}{2}\right) + \frac{1-\sqrt{5}}{5-\sqrt{5}}$$

$$\alpha = \frac{-1-\sqrt{5}}{2\sqrt{5}} = \frac{\varphi - 1}{\sqrt{5}}$$