Calculus and Differential Equations II

MATH 250 B

Linear differential equations: introduction & overview

Definitions

- A linear ordinary differential equation of order n is an equation of the form

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = h(x),$$

  where the coefficients a_i are functions of x.

- If the functions a_i are constant, then the equation is said to have constant coefficients.

- If $h(x) = 0$, then the differential equation is said to be homogeneous.

- A solution to the above differential equation is an n-times differentiable function $y(x)$ which satisfies the differential equation.

Definitions (continued)

- An initial condition is the prescription of the values of y and of its $(n-1)$st derivatives at a point x_0,

$$y(x_0) = y_0,\ \frac{dy}{dx}(x_0) = y_1, \ldots, \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1},$$

  where y_0, y_1, ... y_{n-1} are given numbers.

- Boundary conditions prescribe the values of linear combinations of y and its derivatives at two different values of x.

- We will see methods to solve linear differential equations. Initial or boundary conditions should be imposed after the general solution of a differential equation has been found.

Examples

- Example 1: Consider $y'' - 2y' + y = 0$.
  - What is the order of this differential equation?
  - Are $y_1(x) = e^x$ and $y_2(x) = x\,e^x$ solutions of this differential equation?
  - Are $y_1(x)$ and $y_2(x)$ linearly independent?
- Example 2: Consider $7xy''' + 2y' = \cos(x)$.
  - Is the differential equation linear?
  - What is its order?
  - Is it homogeneous?
  - Does it have constant coefficients?

Existence and uniqueness of solutions

- Theorem: If the functions $a_i(x)$ and $h(x)$ are continuous on the interval (a, b) and if $x_0 \in (a, b)$, then there exists, on (a, b), a unique solution to

$$\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = h(x),$$

that satisfies the initial conditions

$$y(x_0) = y_0,\ \frac{dy}{dx}(x_0) = y_1, \ldots, \frac{d^{n-1}y}{dx^{n-1}}(x_0) = y_{n-1}.$$

- Example: Does the initial value problem $y^{(4)} - x^3 y'' + 3y = 0$, with $y(0) = 1,\ y'(0) = 1,\ y''(0) = 0,\ y^{(3)}(0) = 0$, have a unique solution on the interval $[-1, 1]$?

General facts concerning linear differential equations

- The general solution y to a non-homogeneous linear equation of order n is of the form

$$y(x) = y_h(x) + y_p(x),$$

where $y_h(x)$ is the general solution to the corresponding homogeneous equation and $y_p(x)$ is a particular solution to the non-homogeneous equation.

- The general solution of a homogeneous linear equation of order n with continuous coefficients is a linear combination of n linearly independent solutions.

General method to solve a linear differential equation

As a consequence, solving a linear differential equation of order n with continuous coefficients will involve the following steps:

1. Find n linearly independent solutions to the homogeneous equation.
2. Use the above to write the general solution $y_h(x)$ to the homogeneous equation.
3. Find a particular solution $y_p(x)$ to the non-homogeneous equation.
4. Use the above to write the general solution $y_g(x) = y_h(x) + y_p(x)$ to the linear (non-homogeneous) equation.
5. Impose the boundary or initial conditions, if any.

What we will do

- We will learn methods to
  - Find linearly independent solutions to
    1. Homogeneous equations with constant coefficients
    2. Homogeneous Cauchy-Euler equations.
  - Find linearly independent solutions to more general homogeneous linear equations, assuming we already know one solution.
  - Decide whether the solutions are linearly independent.
  - Find a particular solution to a non-homogeneous linear equation.
- We will work mostly with second order differential equations, but the methods we will learn can easily be generalized to equations of higher order.