Calculus and Differential Equations II

MATH 250 B

Linear systems of differential equations

Second order autonomous linear systems

- We are mostly interested with 2×2 first order autonomous systems of the form

$$\begin{cases} x' = a\,x + b\,y \\ y' = c\,x + d\,y \end{cases}$$

  where x and y are functions of t and a, b, c, and d are real constants.

- Such a system may be re-written in matrix form as

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = M \begin{bmatrix} x \\ y \end{bmatrix}, \qquad M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- The purpose of this section is to classify the dynamics of the solutions of the above system, in terms of the properties of the matrix M.

Existence and uniqueness (general statement)

- Consider a linear system of the form

$$\frac{dY}{dt} = M(t)Y + F(t),$$

where Y and $F(t)$ are $n \times 1$ column vectors, and $M(t)$ is an $n \times n$ matrix whose entries may depend on t.

- Existence and uniqueness theorem: If the entries of the matrix $M(t)$ and of the vector $F(t)$ are continuous on some open interval I containing t_0, then the initial value problem

$$\frac{dY}{dt} = M(t)Y + F(t), \qquad Y(t_0) = Y_0$$

has a unique solution on I.

- In particular, this means that trajectories in the phase space do not cross.

General solution

- The general solution to $Y' = M(t)Y + F(t)$ reads

$$\begin{aligned} Y(t) &= C_1\, Y_1(t) + C_2\, Y_2(t) + \cdots + C_n\, Y_n(t) + Y_p(t), \\ &= U(t)\, C + Y_p(t), \end{aligned}$$

where

  - $Y_p(t)$ is a particular solution to $Y' = M(t)Y + F(t)$.
  - $Y_i(t),\ i = 1, \ldots, n$, are n linearly independent solutions to the associated homogeneous system $Y' = M(t)Y$.
  - The coefficients $C_i,\ i = 1, \ldots, n$, are n arbitrary constants.
  - The matrix $U(t) = [Y_1(t)\ \ Y_2(t)\ \ \cdots\ \ Y_n(t)]$ is called a fundamental matrix of the homogeneous system $Y' = M(t)Y$.
  - The vector C has constant entries $C_1,\ C_2,\ \cdots,\ C_n$.

General solution (continued)

To solve the linear system, we therefore proceed as follows.

1. Find n linearly independent solutions $Y_1(t), \ldots, Y_n(t)$ of the homogeneous system.

2. Write the general solution to the homogeneous system as a linear combination of the Y_i's,
$Y_h(t) = C_1\, Y_1(t) + C_2\, Y_2(t) + \cdots + C_n\, Y_n(t)$.

3. Find a particular solution to the full system, $Y_p(t)$.

4. Write the general solution to the full system,
$Y(t) = Y_h(t) + Y_p(t)$.

5. Impose the initial conditions, if any.

In what follows, we are only concerned with 2×2 homogeneous systems with constant coefficients.

Homogeneous systems with constant coefficients

- The method of solution discussed below is for a 2×2 homogeneous first order system with constant coefficients. It may of course be generalized to systems of higher dimension.
- We look for a solution to $Y' = M\,Y$ in the form

$$Y(t) = \xi\, e^{\lambda t}, \qquad \xi \neq 0.$$

- We find that ξ and λ is an eigenvalue-eigenvector pair of M, i.e. that

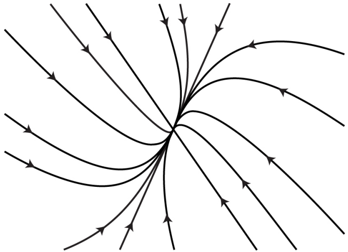
$$M\,\xi = \lambda\,\xi.$$

- The eigenvalue λ must solve the characteristic equation

$$\lambda^2 - \lambda\,\mathrm{Tr}(M) + \det(M) = 0,$$

where $\mathrm{Tr}(M)$ is the trace of M, and $\det(M)$ is its determinant.

Two distinct real roots of the same sign



Trajectories in the phase plane near a sink or stable node.

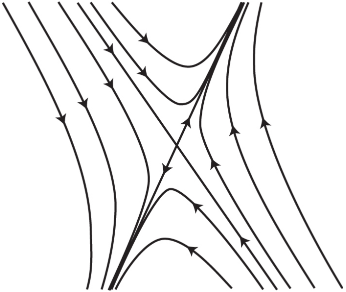
- For two distinct real eigenvalues λ_1 and λ_2 with eigenvectors ξ_1 and ξ_2, we have

$$Y(t) = C_1\,\xi_1\,e^{\lambda_1 t} + C_2\,\xi_2\,e^{\lambda_2 t}.$$

- If λ_1 and λ_2 are both positive, i.e. if $\mathrm{Tr}(M) > 0$, the origin is called a source or an unstable node.
- If λ_1 and λ_2 are both negative, the origin is called a sink or a stable node.
- At the origin, trajectories are tangent to the slower direction.

In this case, $\det(M) > 0$ and $\mathrm{Tr}(M)^2 - 4\det(M) > 0$.

Two real roots of opposite signs

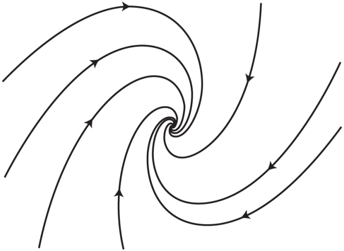


Trajectories in the phase plane near a saddle point.

In this case, $\det(M) < 0$.

- If λ_1 and λ_2 have opposite signs, the origin is called a saddle point.
- If $\lambda_1 < 0$, initial conditions chosen on the line through the origin parallel to ξ_1 converge towards the origin as t increases. This straight line is called the stable manifold of the origin.
- Similarly, the unstable manifold consists of initial conditions that converge towards the origin backwards in time, and is the line through the origin parallel to ξ_2 (associated with $\lambda_2 > 0$).

Two complex conjugate roots



Trajectories in the phase plane near a stable spiral.

- For two complex conjugate eigenvalues λ and $\bar{\lambda}$ with eigenvectors ξ and $\bar{\xi}$, we have

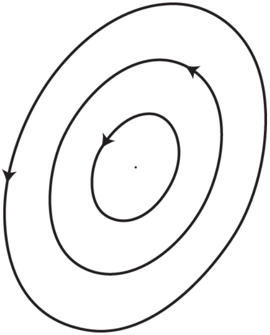
$$Y(t) = C\,\xi\,e^{\lambda t} + \bar{C}\,\bar{\xi}\,e^{\bar{\lambda} t}.$$

- If $\Re e(\lambda) > 0$, the origin is called an unstable spiral.

- If $\Re e(\lambda) < 0$, the origin is called a stable spiral.

In this case, $\mathrm{Tr}(M)^2 - 4\det(M) < 0$.

Two purely imaginary roots



Trajectories in the phase plane near a center.

- For two purely imaginary complex conjugate eigenvalues $\lambda = \pm i\beta$ with eigenvectors ξ and $\bar{\xi}$, we have

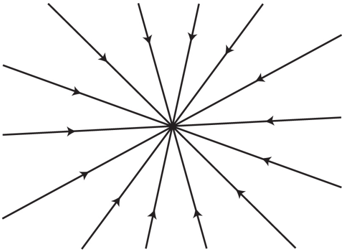
$$Y(t) = C\,\xi\, e^{i\beta t} + \bar{C}\,\bar{\xi}\, e^{-i\beta t}.$$

- The origin is called a center. It is (marginally) stable.

- Trajectories are closed orbits (ellipses) about the origin.

In this case, $\mathrm{Tr}(M) = 0$ and $\mathrm{Tr}(M)^2 - 4\det(M) < 0$.

One real double root - star



Trajectories in the phase plane near a stable star.

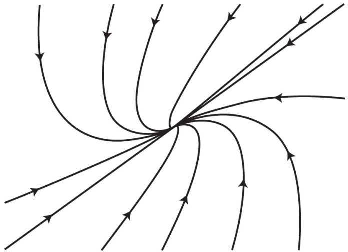
- For two identical eigenvalues $\lambda_1 = \lambda_2 = \lambda$ with two linearly independent eigenvectors ξ_1 and ξ_2, we have

$$Y(t) = C_1\,\xi_1\,e^{\lambda t} + C_2\,\xi_2\,e^{\lambda t}.$$

- If λ is positive, the origin is called an unstable star.
- If λ is negative, the origin is called a stable star.
- All trajectories about a star are straight lines.

In this case, $\det(M) > 0$ and $\mathrm{Tr}(M)^2 - 4\det(M) = 0$.

One real double root - degenerate node



Trajectories in the phase plane near a stable degenerate node.

- For two identical eigenvalues λ with only one eigendirection ξ_1, solutions are of the form

$$Y(t) = C_1\,\xi_1\,e^{\lambda t} + C_2\,(t\xi_1 + \xi_2)\,e^{\lambda t},$$

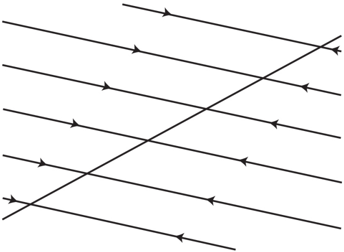
  where ξ_2 is such that $(M - \lambda I)\xi_2 = \xi_1$. It is called a generalized eigenvector of the matrix M.

- If λ is positive, the origin is called an unstable degenerate node.

- If λ is negative, the origin is called a stable degenerate node.

In this case, $\det(M) > 0$ and $\mathrm{Tr}(M)^2 - 4\det(M) = 0$.

One zero root



Trajectories in the phase plane near a stable line of fixed points.

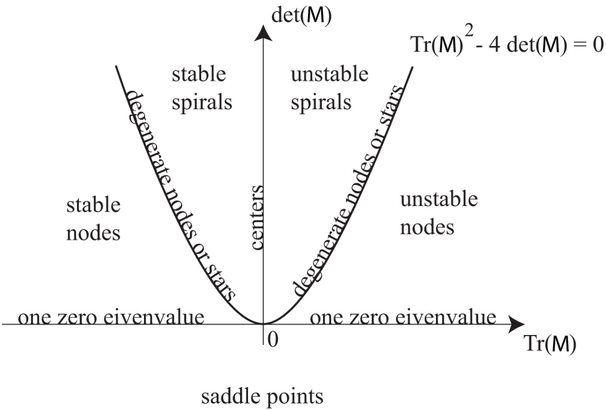
- For $\lambda_1 = 0$ and $\lambda_2 \neq 0$ with associated eigenvectors ξ_1 and ξ_2, solutions are of the form

$$Y(t) = C_1\,\xi_1 + C_2\,\xi_2\,e^{\lambda_2 t}.$$

- Any solution of the form $C_1\,\xi_1$ is constant, i.e. there is a line of fixed points going through the origin and parallel to ξ_1.
- If $\lambda_2 < 0$, this line of fixed points is stable. If $\lambda_2 > 0$, it is unstable.

In this case, $\det(M) = 0$.

Summary



These results may be summarized in the above diagram, which shows how the origin is classified as a function of the trace and determinant of M.

Also see the Linear Phase Portraits applet from MIT.

What comes next?

- MATH 215: Introduction to Linear Algebra
  Vector spaces, linear transformations and matrices.

- MATH 223: Vector Calculus
  Vectors, differential and integral calculus of several variables.

- MATH 322: Mathematical Analysis for Engineers
  Complex functions and integration, line and surface integrals, Fourier series, partial differential equations.

- MATH 454: Ordinary Differential Equations and Stability Theory
  General theory of initial value problems, linear systems and phase portraits, linearization of nonlinear systems, stability and bifurcation theory, an introduction to chaotic dynamics.

And of course

- MATH/MCB/PHYS 303: Explorations in Integrated Science
  Integrates knowledge and research approaches from multiple scientific disciplines through laboratory- and lecture- based modules.

- MATH 363: Introduction to Statistical Methods
  Discusses issues of collection, model derivation and analysis, interpretation, explanation, and presentation of data.

- MATH 485/585: Mathematical Modeling
  Development, analysis, and evaluation of mathematical models for physical, biological, social, and technical problems.

- PSIO 472/572 - Quantitative Modeling of Biological Systems
  Techniques for development of mathematical models for biological phenomena.