Calculus and Differential Equations II

MATH 250 B

Second order equations with constant coefficients

Homogeneous equations

- In this section, we consider equations of the form

$$a\,\frac{d^2y}{dx^2} + b\,\frac{dy}{dx} + c\,y = 0,$$

  where the coefficients a, b, and c are given constants.

- We need to find two linearly independent solutions, $y_1(x)$ and $y_2(x)$ to the above equation.

- Try a solution of the form $y(x) = exp(\lambda x)$.

- Then, λ must be a root of the characteristic equation

$$a\lambda^2 + b\lambda + c = 0.$$

Examples

For each equation below, find two linearly independent solutions, and then solve the differential equation with the given initial conditions.

- Example 1: $y'' + y' - 2y = 0$, with initial conditions $y(0) = 1$, $y'(0) = 0$.

- Example 2: $y'' + 6y' + 25y = 0$, with initial conditions $y(0) = 0$, $y'(0) = 1$.

- Example 3: $y'' + 2y' + y = 0$, with initial conditions $y(0) = 1$, $y'(0) = 1$.

General Method of solution

To find the general solution to an ordinary differential equation of the form $ay'' + by' + cy = 0$, where $a, b, c \in \mathbb{R}$, proceed as follows.

1. Find the characteristic equation, $a\lambda^2 + b\lambda + c = 0$ and solve for the roots λ_1 and λ_2.
2. If $b^2 - 4ac > 0$, then the two roots are real and the general solution is $y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.
3. If $b^2 - 4ac < 0$ the two roots $\alpha \pm i\beta$ are complex conjugate of one another, with $\alpha = \frac{-b}{2a}$ and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$. The general solution is of the form $y = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right)$.
4. If $b^2 - 4ac = 0$, then there is a double root $\lambda = -\frac{b}{2a}$, and the general solution is $y = \left(C_1 + C_2\, x\right) e^{\lambda x}$.

Non-homogeneous equations

- Recall that we only need to find one particular solution to a non-homogenous linear equation.
- Such a solution may be found
  1. By inspection.
  2. By the method of variation of parameters.
  3. By the method of undetermined coefficients.
- The method of variation of parameters is general and may be applied to all linear ordinary differential equations.
- The method of undetermined coefficients is specific to linear equations with constant coefficients.

Variation of parameters

- This method assumes that we already know the general solution to the homogeneous equation. Assume this solution is of the form $y_h(x) = C_1\, y_1(x) + C_2\, y_2(x)$, where C_1 and C_2 are constants and y_1 and y_2 are two linearly independent solutions to the homogeneous equation.
- Look for a particular solution to the non-homogeneous equation in the form $y_p(x) = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$. In other words, the constants are now varying.
- The idea is to substitute this solution into the non-homogeneous equation, and solve for $u_1(x)$ and $u_2(x)$.
- Since there are two functions to find, we need a second equation, which we set to $u_1'(x)y_1(x) + u_2'(x)y_2(x) = 0$.

Variation of parameters (continued)

- We then have to solve the 2×2 system for u_1' and u_2',

$$\begin{cases} u_1' y_1 + u_2 y_2' = 0 \\ u_1' y_1' + u_2' y_2' = \frac{f(x)}{a(x)} \end{cases},$$

  for a non-homogeneous linear equation of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x).$$

- The determinant of this system, which is called the Wronskian of y_1 and y_2,

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix},$$

  is non-zero since y_1 and y_2 are two linearly independent solutions of the homogeneous equation.

Variation of parameters (continued)

- As a consequence, one can always solve for u_1' and u_2'.
- Integration gives u_1 and u_2, which are then substituted back in the expression for y_p to obtain a particular solution.
- Example 1: Use the method of variation of parameters to show that a particular solution to $y'' + y' - 2y = x$ is

$$y_p(x) = -\frac{x}{2} - \frac{1}{4}.$$

- Example 2: Use the method of variation of parameters to show that a particular solution to $y'' + 6y' + 25y = \cos(4x)$ is

$$y_p(x) = \frac{1}{73}\left(\cos(4x) + \frac{8}{3}\sin(4x)\right).$$

Method of undetermined coefficients

- This method may be used to find particular solutions to linear equations with constant coefficients, and with a non-homogeneous term of the form

$$f(x) = P_m(x)\exp(\alpha x)\cos(\beta x) + Q_m(x)\exp(\alpha x)\sin(\beta x),$$

where P_m and Q_m are polynomials in x of degree m.

- The idea of the method is as follows.
  1. Start with a trial solution y_p with the right functional form but undetermined coefficients;
  2. Substitute y_p into the differential equation;
  3. Solve for the coefficients.

Method of undetermined coefficients (continued)

The trial function y_p is found as follows.

- If $\alpha \pm i\beta$ are not roots of the characteristic equation, try

  $$y_p(x) = K_m(x)\exp(\alpha x)\cos(\beta x) + L_m(x)\exp(\alpha x)\sin(\beta x),$$

  where K_m and L_m are polynomials of degree m.

- If $\alpha \pm i\beta$ are roots of the characteristic equation of multiplicity h, then try

  $$y_p(x) = x^h\left(K_m(x)\exp(\alpha x)\cos(\beta x) + L_m(x)\exp(\alpha x)\sin(\beta x)\right),$$

  where K_m and L_m are polynomials of degree m.

Method of undetermined coefficients (continued)

- Example 1: Use the method of undetermined coefficients to find a particular solution to

$$y'' - 4y' + 4y = 12x\exp(2x).$$

- Example 2: Use the method of undetermined coefficients to find a particular solution to

$$y'' + 6y' + 25y = \cos(4x).$$

- If the non-homogeneous term involves linear combinations of sines and cosines with polynomial coefficients, then use the the principle of superposition (see next page).

  Example 3: Use the method of undetermined coefficients to find a particular solution to

$$y'' + 6y' + 25y = \cos(4x) + x.$$

Principle of superposition (general case)

- If y_{p1} solves a linear equation of the form

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f_1(x),$$

and if y_{p2} solves the same equation with a different right-hand side,

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f_2(x),$$

then $y_p = y_{p1} + y_{p2}$ solves

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f_1(x) + f_2(x).$$

- As a consequence, to find a particular solution y_p to the last equation, it may be easier to solve for the first two equations separately, and then write y_p as the sum of y_{p1} and y_{p2}.