Calculus and Differential Equations II

MATH 250 B

Sequences and series

Sequences

- A sequence is an infinite list of numbers, $s_1, s_2, \ldots, s_n, \ldots,$ indexed by integers.
- Example 1: Find the first five terms of $s_n = (-1)^n \left(\frac{1}{3}\right)^n$, $n \geq 1$.
- Example 2: Find a formula for s_n, $n \geq 1$, given that its first five terms are $0,\ 2,\ 6,\ 14,\ 30$.
- Some sequences are defined recursively. For instance, $s_n = 2\, s_{n-1} + 3$, $n > 1$, with $s_1 = 1$.
- If $\lim_{n\to\infty} s_n = L$, where L is a number, we say that the sequence (s_n) converges to L. If such a limit does not exist or if $L = \pm\infty$, one says that the sequence diverges.

Sequences (continued)

- Example 3: Does the sequence $\left(\frac{2^n}{5^n}\right)$ converge?
  1. Yes
  2. No
- Example 4: Does the sequence $\left(\frac{n}{2}+\frac{5}{n}\right)$ converge?
  1. Yes
  2. No
- Example 5: Does the sequence $\left(\frac{\sin(2n)}{n}\right)$ converge?
- Remarks:
  1. A convergent sequence is bounded, i.e. one can find two numbers M and N such that $M < s_n < N$, for all n's.
  2. If a sequence is bounded and monotone, then it converges.

Series

- A series is a pair of sequences, (S_n) and (u_n) such that

$$S_n = \sum_{k=1}^{n} u_k.$$

- A geometric series is of the form

$$S_n = a + ax + ax^2 + ax^3 + \cdots + ax^{n-1}, \qquad u_k = ax^{k-1}$$

- One can show that if $x \neq 1$, $S_n = a\dfrac{1-x^n}{1-x}$.

- Example 1: Is $2 + 1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \ldots$ geometric?
  1. Yes
  2. No

Geometric series

- Example 2: Is $1 + x + 2x^2 + 3x^3 + 4x^4 + \ldots$ geometric?
  1. Yes
  2. No
- Example 3: Find $3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots + 3(0.1)^{10}$.
- Example 4: Assume $\sum_{k=1}^{n} u_k$ is geometric and that $u_k \neq 0$ for all k's. Is $\sum_{k=1}^{n} \frac{1}{u_k}$ geometric as well?
  1. Yes
  2. No

Convergence of series

- Recall that we defined the partial sum S_n as $S_n = \sum_{k=1}^{n} u_k$.
- If S_n converges and $\lim_{n\to\infty} S_n = S$, then one says that the infinite series $\sum_{k=1}^{\infty} u_k$ converges, and that its sum is equal to S.

  One writes
$$\sum_{k=1}^{\infty} u_k = S.$$
- If $\lim_{n\to\infty} S_n$ does not exist or is infinite, one says that the series $\sum_{k=1}^{\infty} u_k$ diverges.

Convergence of geometric series

- Recall that an infinite geometric series is written as an infinite sum of the form

$$a + ax + ax^2 + \cdots + ax^n + \ldots \qquad a,\, x \neq 0$$

- If $|x| < 1$, the above series converges to $\dfrac{a}{1-x}$.
- If $|x| > 1$, the series diverges.
- If $x = 1$, $\displaystyle\sum_{k=1}^{n} a = na$ and the series diverges.
- If $x = -1$, $\displaystyle\sum_{k=1}^{n} a = a - a + a - a + \ldots$, and the series diverges.

More on convergence of series

- If $\sum_{n=1}^{n} a_n$ and $\sum_{n=1}^{n} b_n$ converge, and if k is a constant, then
  1. $\sum_{n=1}^{n}(a_n + b_n)$ converges to $\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$
  2. $\sum_{n=1}^{n} k\, a_n$ converges to $k \sum_{n=1}^{\infty} a_n$.
- Changing a finite number of terms in a series does not change its convergence properties.
- If $\lim_{n\to\infty} a_n$ is not zero, or if it does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} k\, a_n$, for $k \neq 0$.

Comparison of series with integrals

- There is a direct correspondence between convergence of series and convergence of integrals. This is called the integral convergence test.
- Assume that $a_n = f(n)$, where f is a continuous, decreasing function that remains positive for $x > b$. Then,
  - If $\displaystyle\int_b^\infty f(x)\,dx$ converges, so does $\displaystyle\sum_{n=1}^{n} a_n$.
  - If $\displaystyle\int_b^\infty f(x)\,dx$ diverges, so does $\displaystyle\sum_{n=1}^{n} a_n$.
- This can be understood by writing the series as a left or right sum approximation of the improper integral.

Comparison of series with integrals (continued)

- Example 1: Does $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converge?
  1. Yes
  2. No

- Example 2: Does $\sum_{n=1}^{\infty} \frac{4}{(2n+3)^3}$ converge?
  1. Yes
  2. No

- Example 3: Does $\sum_{n=1}^{\infty} \frac{1}{1+n}$ converge?
  1. Yes
  2. No

Tests for convergence

- Theorem 1: The Comparison Test
  Suppose $0 \le a_n \le b_n$ for all n's. Then,
  1. If $\sum_{n=1}^{\infty} b_n$ converges, so does $\sum_{n=1}^{\infty} a_n$.
  2. If $\sum_{n=1}^{\infty} a_n$ diverges, so does $\sum_{n=1}^{\infty} b_n$.

- Theorem 2: Suppose $a_n > 0$ and $b_n > 0$ for all n's. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \qquad c > 0,$$

then the two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

Tests for convergence (continued)

- Definitions:
  1. The series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ converges.
  2. The series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if $\sum_{n=1}^{\infty} |a_n|$ diverges but $\sum_{n=1}^{\infty} a_n$ converges.
- Theorem 3: Absolute convergence implies convergence
  If $\sum_{n=1}^{\infty} |a_n|$ converges, so does $\sum_{n=1}^{\infty} a_n$.

Tests for convergence (continued)

- Theorem 4: The Ratio Test

  Consider the series $\sum_{n=1}^{\infty} a_n$ and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then,

1. If $L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, more information is needed to conclude.

Tests for convergence (continued)

- Theorem 5: The Alternating Series Test

  Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ and suppose that

$$\lim_{n\to\infty} a_n = 0, \qquad \text{and} \qquad 0 < a_{n+1} < a_n.$$

  Then, the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ converges.

- Theorem 6: Error bounds for alternating series

  Assume that the conditions of Theorem 5 are satisfied, and let

$$S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n, \qquad S_n = \sum_{k=1}^{n}(-1)^{k-1}a_k.$$

  Then, $|S - S_n| < a_{n+1}$ for all n's.

Example

- Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

  Can we use the ratio test to conclude convergence?

  1. Yes
  2. No

- Can we use the alternating series test instead?

  1. Yes
  2. No

- The series is

  1. Conditionally convergent
  2. Absolutely convergent

- How many terms should we include in the partial sum S_n to guarantee that S_n is an approximation of S correct to 3 decimal places?

More examples

- Does the series $\displaystyle\sum_{n=0}^{\infty} \frac{2}{\sqrt{2+n}}$ converge?
  1. Yes
  2. No
- Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{n+2^n}{n\,2^n}$ converge?
  1. Yes
  2. No
- Does the series $\displaystyle\sum_{n=5}^{\infty} \frac{n\;2^n}{3^n}$ converge?
  1. Yes
  2. No
- Does the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\,2^n}{n^2}$ converge?
  1. Yes
  2. No

Power series

- A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \ldots$$

- Abel's lemma: If the power series converges for $|x-a| = R_0$, then it converges absolutely for all x's such that $|x-a| < R_0$.
- As a consequence, one can define its radius of convergence R:
  - If the series only converges for $x = a$, one says that $R = 0$.
  - If the series converges for all x's, one says that $R = \infty$.
  - Otherwise, R is the largest number such that the series converges for $|x-a| < R$.
- The interval of convergence of the power series is $(a-R, a+R)$ plus any end point where the series converges.

Power series (continued)

- Example 1: What are the radius of convergence and the interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$?

- To find the radius of convergence of a power series, use the ratio test.

- Example 2: Find the interval of convergence of the power series $\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

- Example 3: Find the interval of convergence of the power series $\sum_{n=1}^{\infty} (-1)^n \frac{(x-3)^n}{2^n \; n^2}$.

Review

- Which of the following sequences are monotone and bounded?

$$I.\ s_n = 10 - \frac{1}{n} \quad II.\ s_n = \frac{10n+1}{n} \quad III.\ s_n = \cos(n) \quad IV.\ s_n = \ln(n)$$

1. I only
2. I and II
3. II and IV
4. I, II, and III

- Which of the following sequences converge?

$$I.\ s_n = \frac{\cos(n)}{n} \quad II.\ s_n = \cos\left(\frac{1}{n}\right) \quad III.\ s_n = n\, e^{-n} \quad IV.\ s_n = \frac{\sin(1/n)}{1/n}$$

1. I and III
2. I, II, and III
3. II and IV
4. All of them

- Which of the following geometric series converge?

$$
\begin{aligned}
I. &\quad 20 - 10 + 5 - 2.5 + \dots \\
II. &\quad 1 - 1.1 + 1.21 - 1.331 + \dots \\
III. &\quad 1 + 1.1 + 1.21 + 1.331 + \dots \\
IV. &\quad 1 + y^2 + y^4 + y^6 \dots \qquad -1 < y < 1
\end{aligned}
$$

1. I only
2. IV only
3. I and IV
4. II and IV
5. None of them

Review (continued)

- Which of the following statements are true?

  (a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=0}^{\infty} a_n$ converges.

  (b) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=0}^{\infty} a_n$ diverges.

  (c) If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

1. (a) and (c) only
2. (a), (b), and (c)
3. (a) and (b) only
4. (b) and (c) only

Review (continued)

- Suppose that the series $\sum_{n=5}^{\infty} c_n x^n$ converges for $x = -5$ and diverges for $x = 8$. Which of the following statements are certainly true?
  (a) The series converges when $x = 12$.
  (b) The series converges when $x = 1$.
  (c) The series diverges when $x = 3$.
  (d) The series diverges when $x = 7$.

1. (a) and (c) only
2. (b) only
3. (a), (c) and (d) only
4. All of them