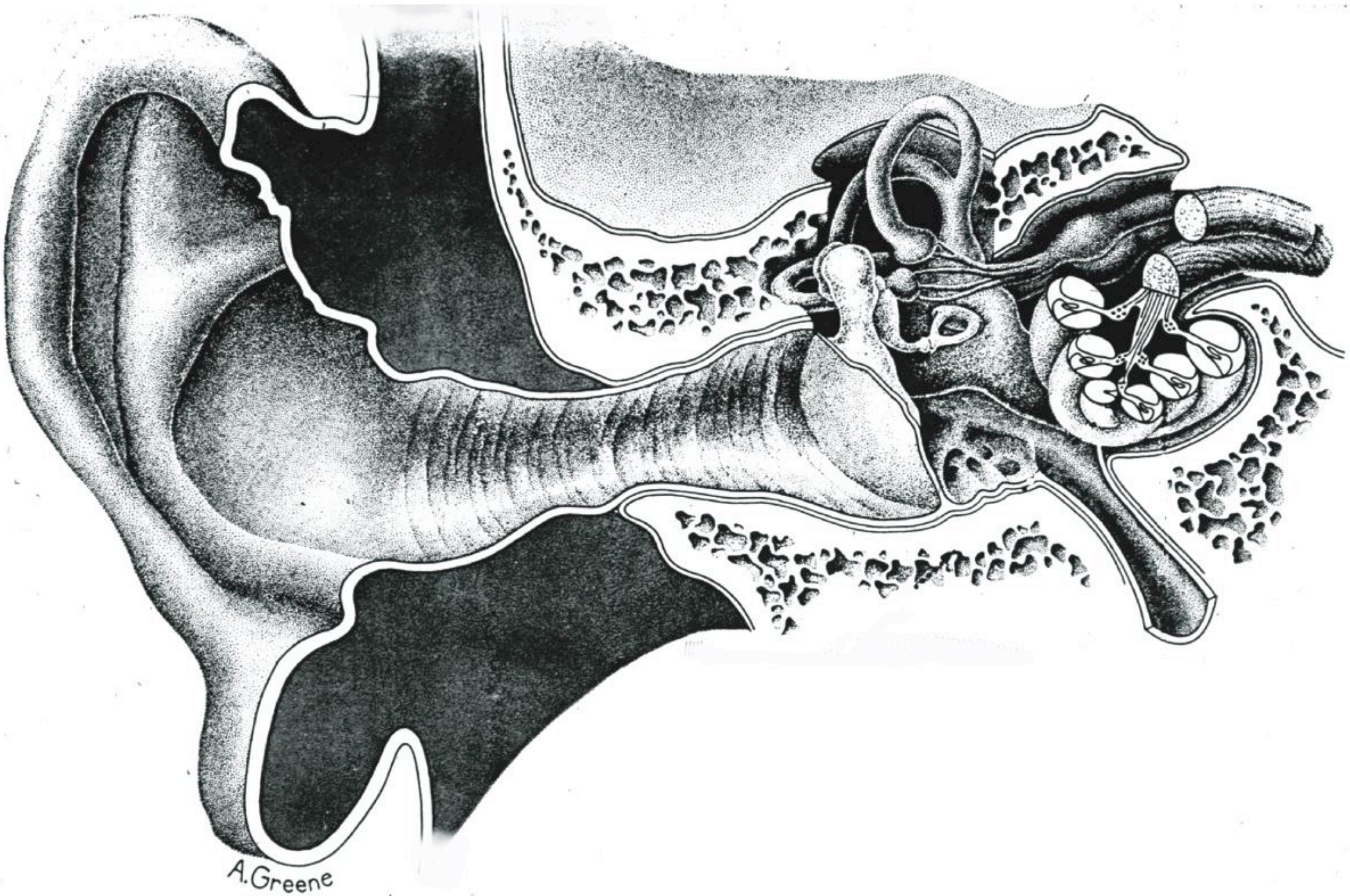
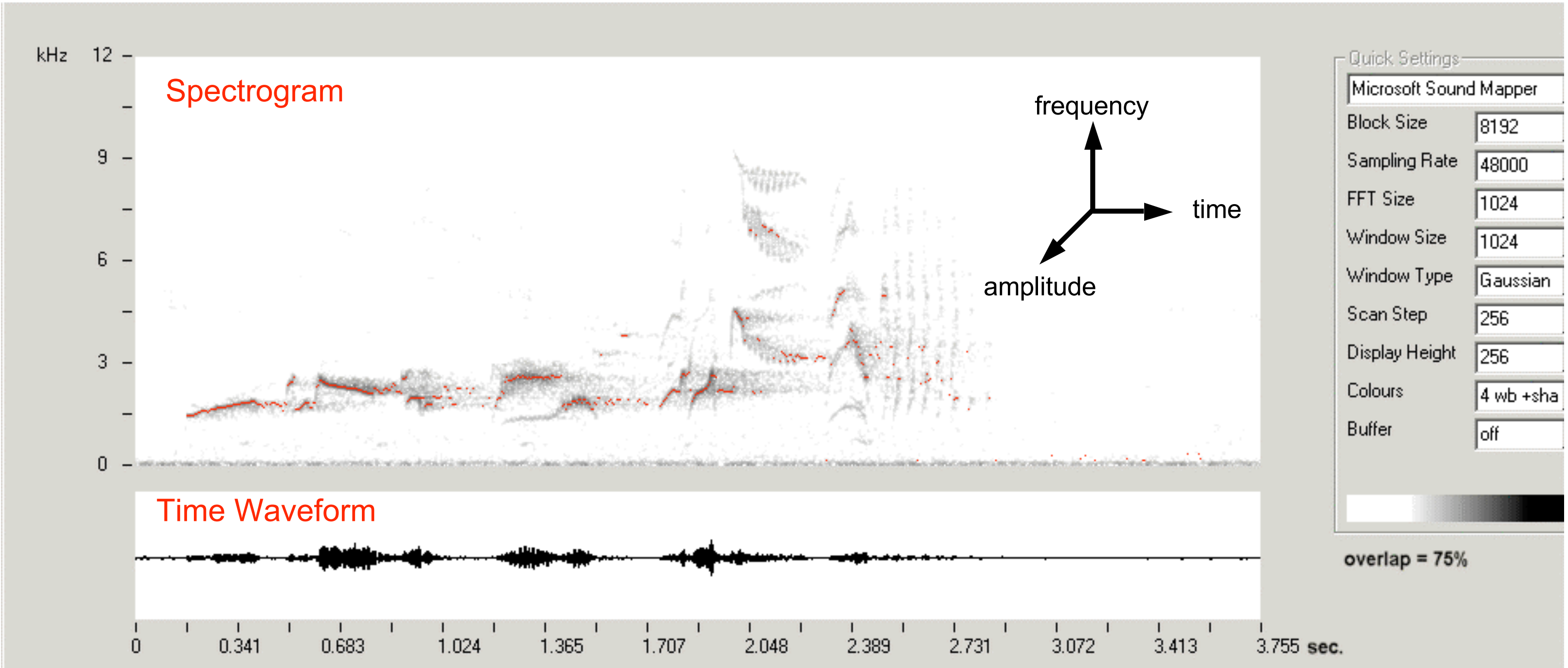
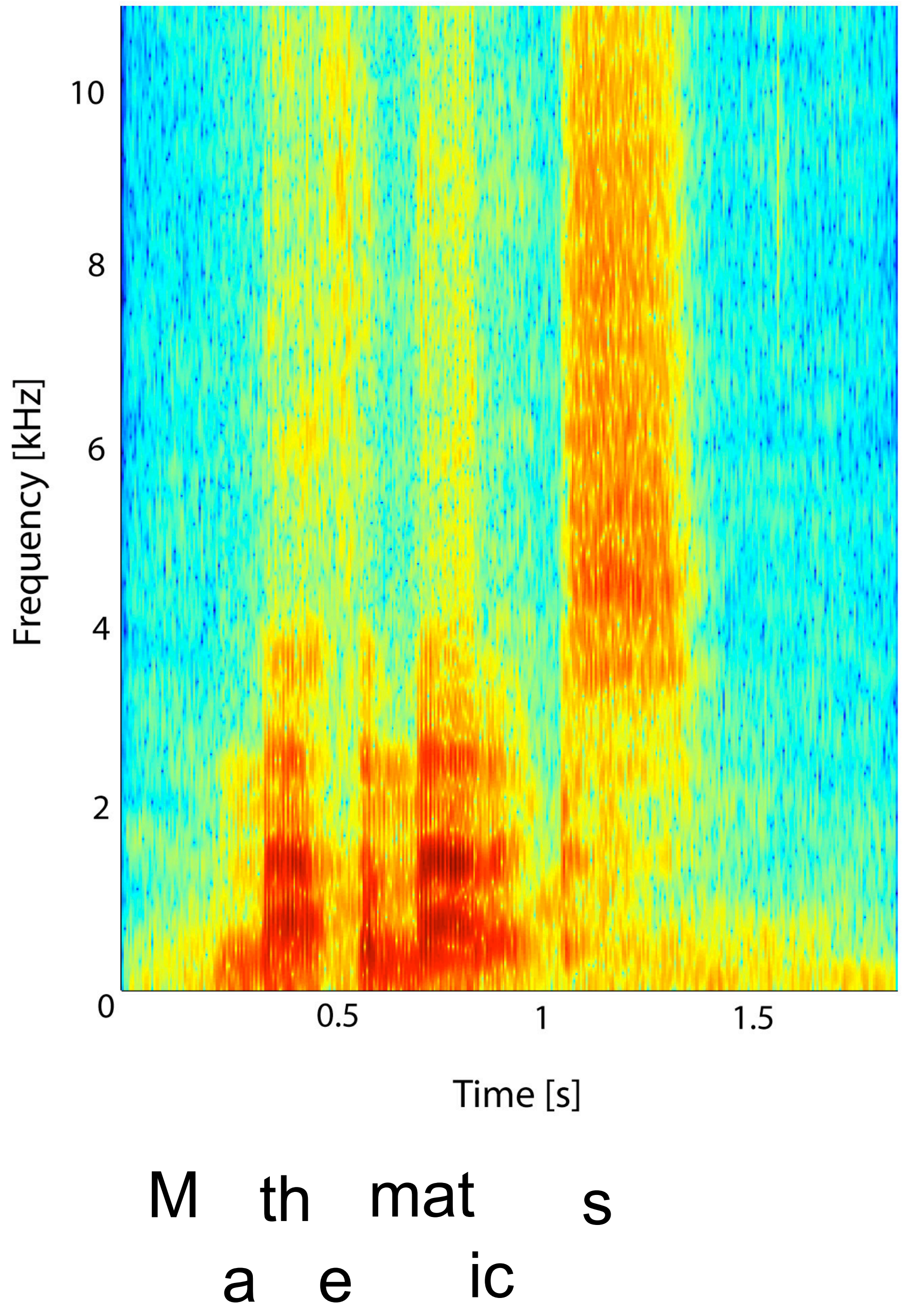
Math 250B (Spring 2009)
03.03.09
C. Bergevin


A.Greene

Blackbird (*Turdus merula*)



http://www.birdsongs.it/



Frequency [kHz]
10
8
6
4
2
0
0.5
1
1.5
Time [s]
M a th e mat ic s

Mathematics: Relationship Between Taylor and Fourier Series

Imagine a periodic time-series (w/ period 2π) described by the following function:

$$f(t) = f(0) + f'(0)t + f''(0)\frac{t^2}{2} + f^{(3)}(0)\frac{t^3}{6} + \cdots = \sum_{n=0}^{\infty} f^{(n)}(0)\frac{t^n}{n!}$$

Taylor series about t=0

OR

$$\approx a_0 + a_1 \cos t + a_2 \cos 2t + a_3 \cos 3t + \cdots$$
$$+ b_1 \sin t + b_2 \sin 2t + b_3 \sin 3t + \cdots$$

Fourier series for t=[-π,π]

$$= a_0 + \sum_{n=1}^{\infty} a_k \cos nt + \sum_{n=1}^{\infty} b_k \sin nt$$

- Taylor series *expands* as a linear combination of polynomials

- Fourier series *expands* as a linear combination of sinusoids

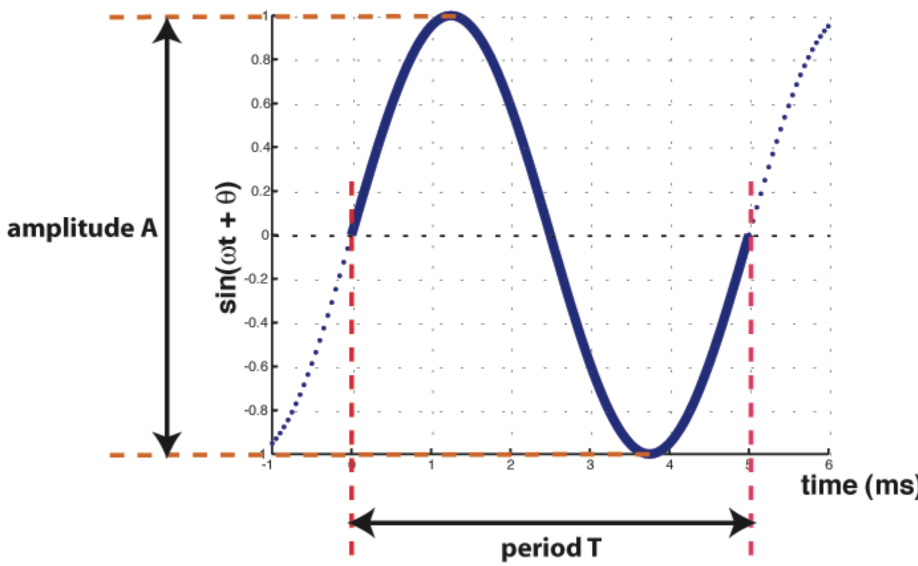
Trigonometry review ⇒ Sinusoids (e.g. tones)

A sinusoid has 3 basic properties:

i. Amplitude - height of wave

ii. Frequency = 1/T [Hz]

iii. Phase - tells you where the peak is (needs a reference)



Why Use Fourier Series?

0. Idea put forth by Joseph Fourier (early 19'th century); his thesis committee was not impressed [though Fourier methods have revolutionized many fields of science and engineering]

1. Many phenomena in nature repeat themselves (e.g., heartbeat, songbird singing)

$\Rightarrow$ Might make sense to 'approximate them by periodic functions'

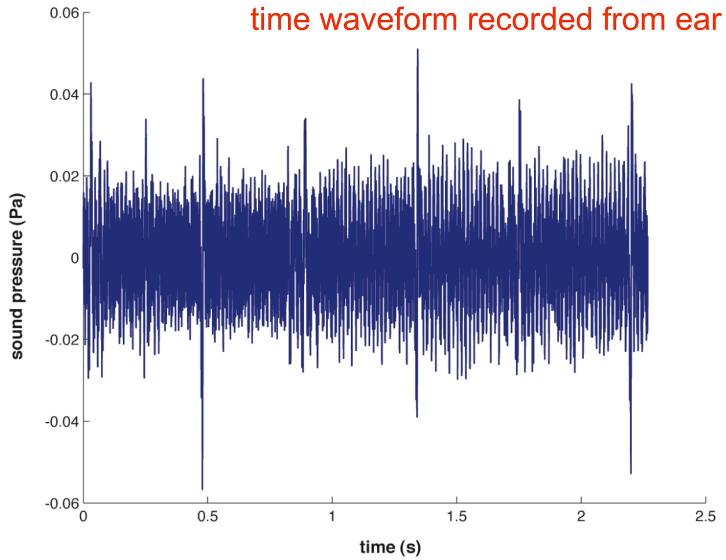
2. Taylor series can give a good *local* approximation (given you are within the radius of convergence); Fourier series give good *global* approximations

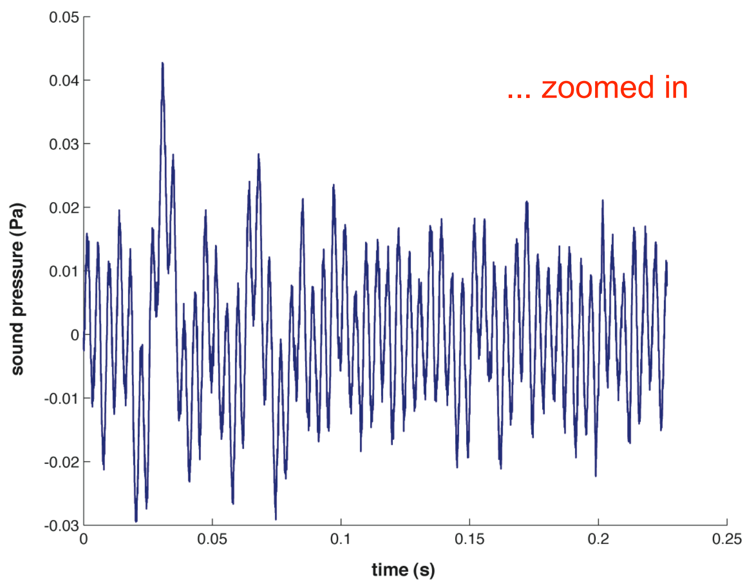
3. Still works even if $f(t)$ is not periodic

4. Fourier series gives us a means to ***transform*** from the time domain to frequency domain and vice versa (e.g., via the FFT)

$\Rightarrow$ Can be easier to see things in one domain as opposed to another

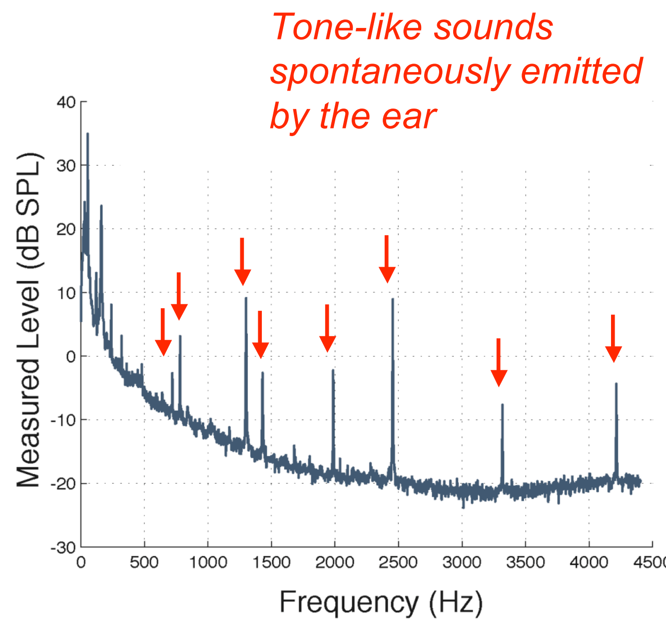
Time Domain





Fourier transform

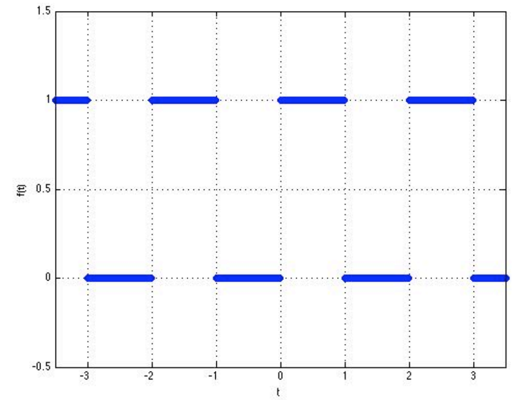
Spectral Domain



⇒ One of the ear's primary functions is to act as a Fourier 'transformer'

<u>Example:</u> Square Wave

$$f(t) = \begin{cases} 0 & -1 < t \le 0 \\ 1 & 0 < t \le 1 \end{cases}$$



For periodic function f with period b, Fourier series on t =[-b/2, b/2] is:

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi k t}{b}\right) + b_k \sin\left(\frac{2\pi k t}{b}\right) \right]$$

where

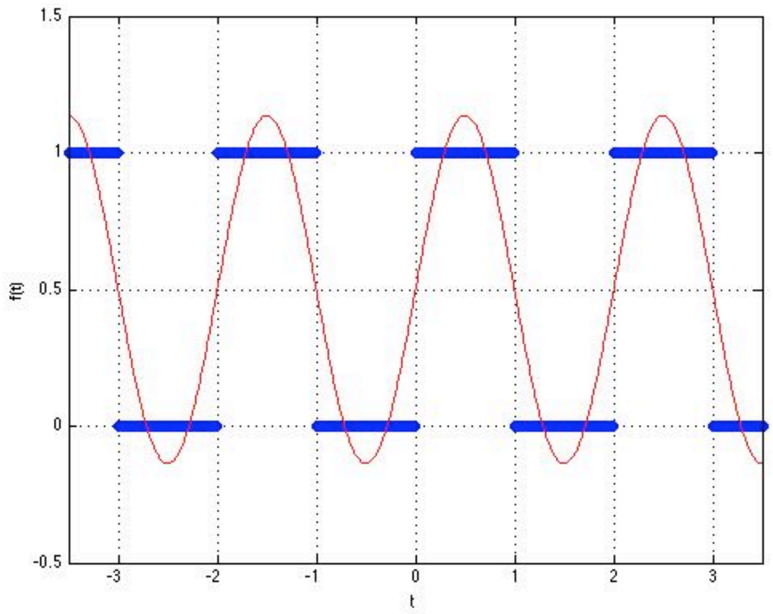
$$a_k = \frac{2}{b} \int_{-b/2}^{b/2} f(t) \cos\left(\frac{2\pi k t}{b}\right) dt \qquad b_k = \frac{2}{b} \int_{-b/2}^{b/2} f(t) \sin\left(\frac{2\pi k t}{b}\right) dt$$

(these are called the *Fourier coefficients*)

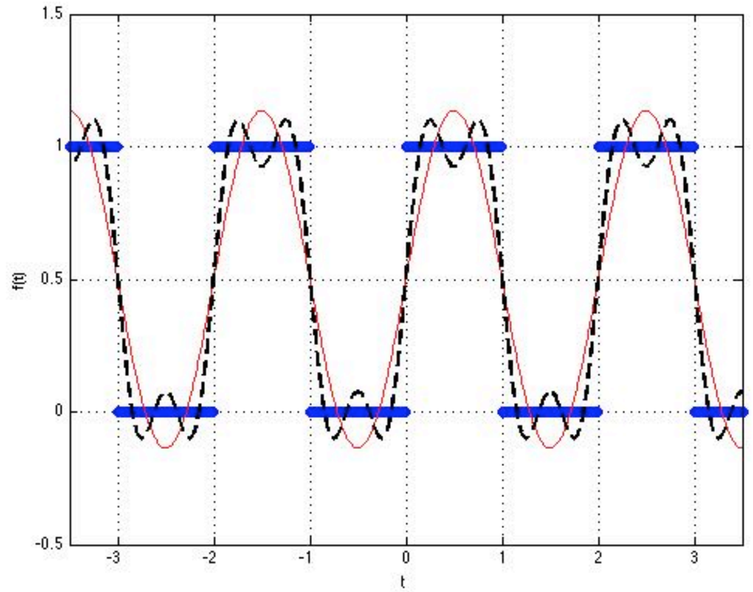
Example: Square Wave (cont.)

⇒ When the smoke clears....

$$f(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\pi t) + \frac{2}{3\pi}\sin(3\pi t) + \frac{2}{5\pi}\sin(5\pi t) + \cdots$$

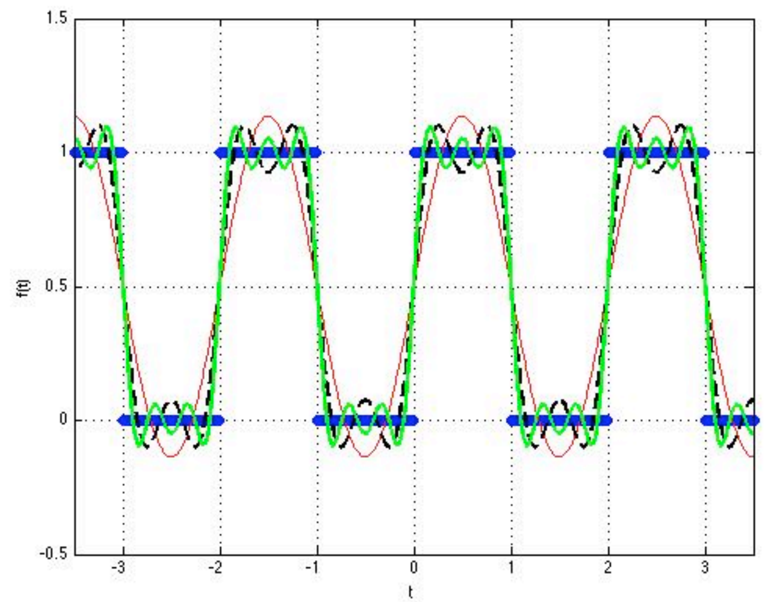


include first two terms only (red)



include first three terms only (black dashed)

include first four terms only (green)



⇒ Note that approximation gets better as the number of higher order terms included increases

SUMMARY

- Taylor series *expands* as a linear combination of polynomials

- Fourier series *expands* as a linear combination of sinusoids

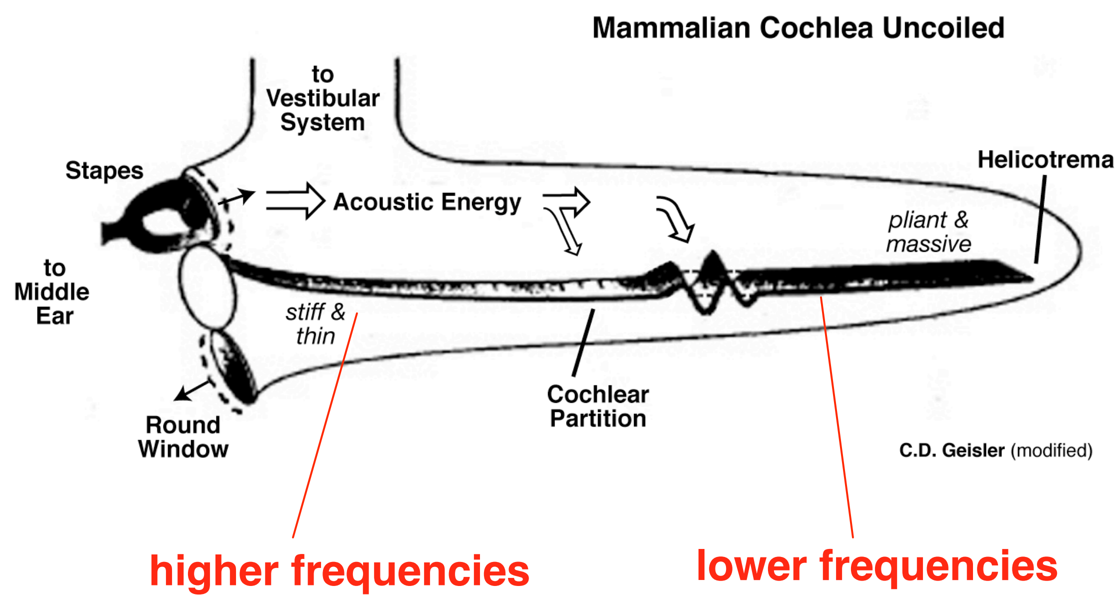
- Idea is that a function (or a time waveform) can effectively be represented as a linear combination of *basis functions*, which can be very useful in a number of different practical contexts

Fini

The ear actually **EMITS** sound!

http://www6.miami.edu/UMH/CDA/UMH_Main/

BM Traveling Waves



- Stimulus induces propagating wave along flexible membrane
- Tonotopic organization (i.e. a spectrum analyzer)
  ⇒ energy *propagates* to its characteristic frequency spot
- Membrane motion stimulates the sensory cells