damping supports motion and pushes with the force

$y'' - 5y' + 6y = 0$

Problem 1

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 3)(\lambda - 2) = 0$

$\lambda = 2,\ \lambda = 3$

$y_h(t) = C_1 e^{3t} + C_2 e^{2t}$

$C_1 = 1 \quad C_2 = 0$

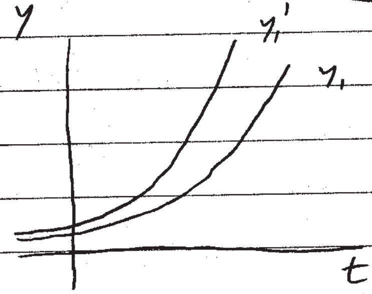
$\Rightarrow y_1(t) = e^{3t}$

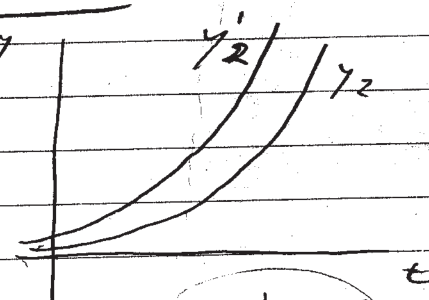
$y_1'(t) = 3e^{3t}$

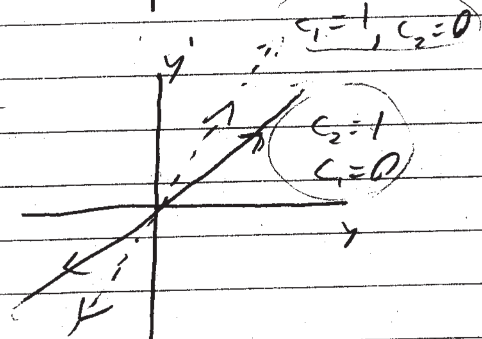
$C_1 = 0$

$C_2 = 1 \Rightarrow y_2(t) = e^{2t}$

$y_2'(t) = 2e^{2t}$







but for other values of C_1 and C_2, if the point on the phase plane is not contained between the two solutions, then the graph curves away from the origin in a clockwise direction.

$e^{2t} = f(c_1, c_2, y, v)$

$e^{3t} = g(c_1, c_2, y, v)$

so $e^t = f^{1/2} = g^{1/3}$ which is why a solution would curve

more on pg 3

NO DAMPING

Problem 2

$y'' + 9y = 0 \qquad y(0)=1 \quad y'(0)=3$

$\lambda^2 + 9 = 0 \qquad C_1 = 1, \quad C_2 = 1$

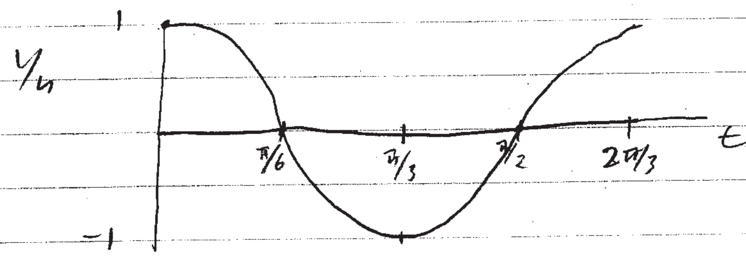
$\lambda = \pm 3i$

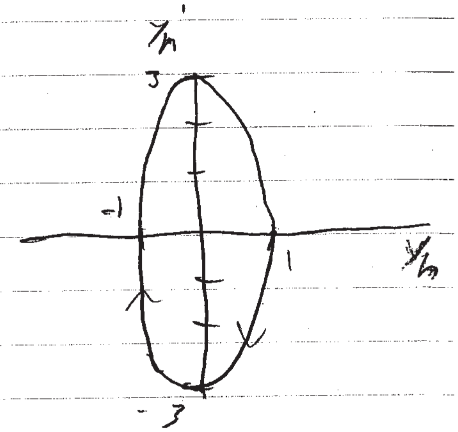
$y_h(t) = C_1 \cos(3t) + C_2 \sin(3t)$

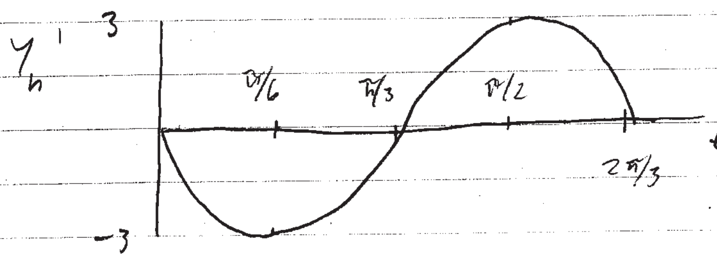
$$\boxed{\begin{aligned} y_h(t) &= \cos(3t) + \sin(3t) \\ y_h'(t) &= -3\sin(3t) + 3\cos(3t) \end{aligned}}$$

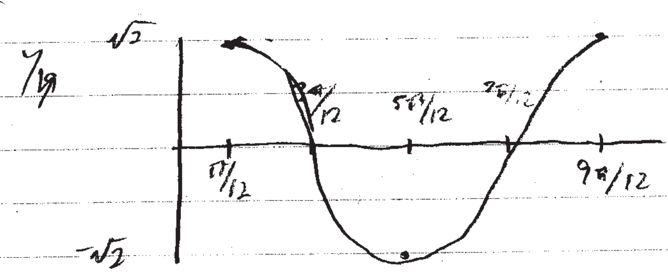
$C_1 = 1, \quad C_2 = 0 \qquad y_h = \cos(3t)$

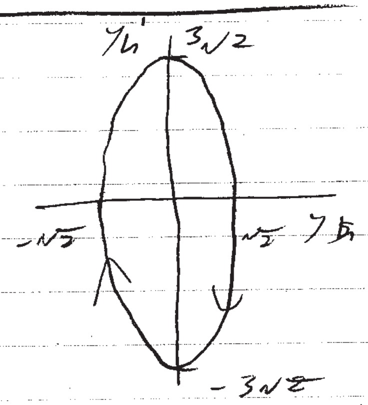
$y_h' = -3\sin(3t)$











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Summary of Problem 1

We can look at the damping properties of each of the four of these problems. Problem 1 has a negative coefficient in front of the second term, which would be the level of damping in, for example, an oscillating system. In both sets of initial conditions, we have graphs that continuously increase as $t \to \infty$, which would indicate a system that has damping that favors the motion of the system. See problem 3

Summary of Problem 2

Problem 2 is typical of a simple harmonic oscillator, which has no damping. We can plot the general solutions, which are sinusoidal. When a plot of y' vs y is drawn, an **ellipse** is the result, which continuously traces it self. This shows an oscillating system that neither speeds up nor slows down.

- Since both y and y' are periodic functions, you get a closed curve, an ellipse, which is also called a <u>center</u>

Nancy Leo
Anthony
Aishah

Discussion Session - April 21st, 2009 #3

Introduction:

Given the 2nd order, linear differential equation:

$$ay'' + by' + cy = 0$$

we find y'' in terms of y' by substituting in a new variable "v" as follows.

$$y'' = -\frac{b}{a}y' - \frac{c}{a}y$$

$$y'' = v' \qquad y' = v$$

$$\left\{\begin{array}{l} y' = v \\ v' = y'' = -\frac{b}{a}v - \frac{c}{a}y \end{array}\right\}$$ we find the following first order system of equations.

Next, given the differential equation of form:

$$y'' - 6y' + 10y = 0$$

We were asked to find
1. General solution
2. Choose initial conditions & plot trajectories in (y, v) plane
3. Generalize the condition/situation.

1. General Solution

$$y'' - 6y' + 10y = 0$$

→ Determine the characteristic Equation

$$\lambda^2 - 6\lambda + 10 = 0$$

* Solve for roots (utilizing quadratic formula).

$$\lambda = 3 \pm i$$

General solution:

$$\boxed{y_n(t) = e^{3t}\left(C_1 \cos(t) + C_2 \sin(t)\right)}$$

Choosing different initial cond. correspond to different values of C_1 & C_2

→ Let $C_1 = C_2 = 0$

$$y_n(t) = e^{3t}(0 + 0) = 0 \quad y_n'(t) = 0$$

→ Let $C_1 = 0$ $C_2 = 1$

$$y_n(t) = e^{3t}(\cos(t) + \cancelto{0}{0\sin(t)})$$

$$y_n'(t) = -e^{3t}\sin(t) + 3e^{3t}\cos(t)$$

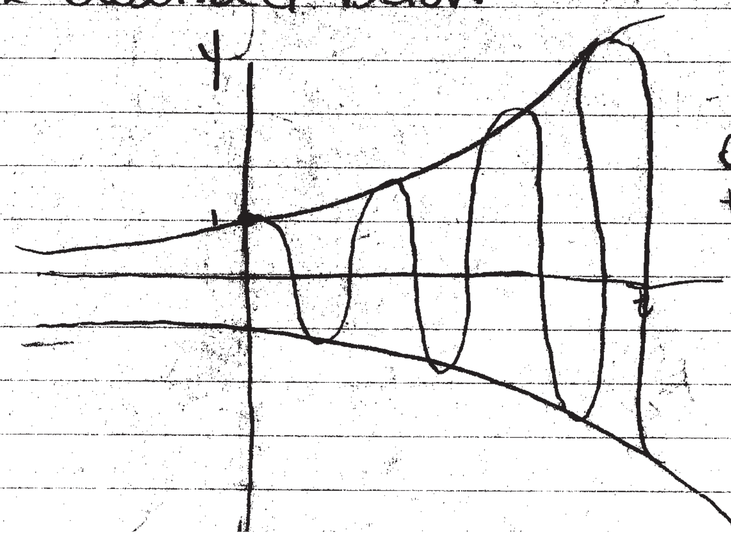
$$= e^{3t}(3\cos(t) - \sin(t))$$

Building an intuition

Now, we generalize the entire situation to help better understand how we can plot the trajectories of the above solution and it's derivative in a (y, y') or (y, v) plane.

- Assume you have:

$y = e^{3t}\cos(t)$ whose graph is the one described below:



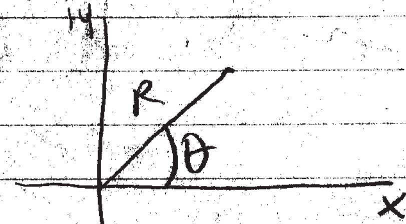
- Here we see $\cos(t)$ osculates between the bounds of e^{3t}, increasing in oscillation as $t \to \infty$ (same as POSITIVE DAMPING).

Now assume you have:

$$y = e^{3t}\cos(t)$$
$$y'' = -e^{3t}\sin(t)$$

- These equations have a form similar to

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$



where e^{3t} behaves like the term that changes the magnitude of the R.

We can now apply this to our case, determining the relationship between $y_n(t) = e^{3t}(C_1\cos(t) + C_2\sin(t))$ and it's derivative on a (y, y') plane).

- The general solution to the differential equation of the form $y'' - 6y' + 10y = 0$ is:

$$y_n(t) = e^{3t}(C_1\cos(t) + C_2\sin(t))$$

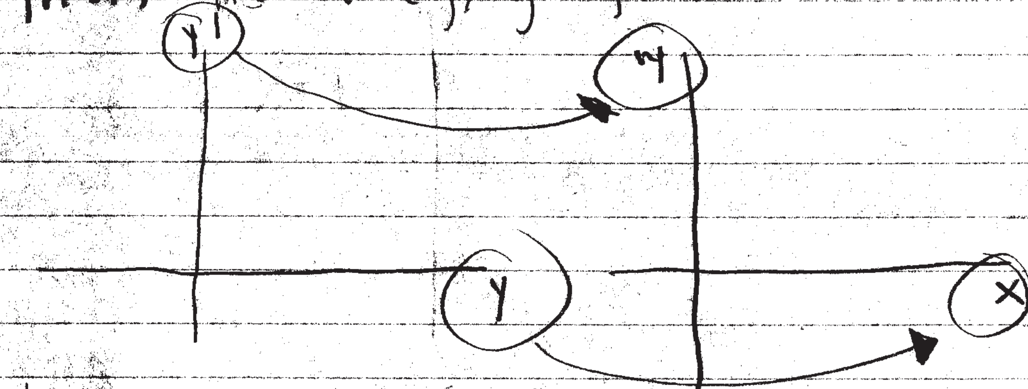
where...

$$y_n'(t) = e^{3t}(-C_1\sin(t) + C_2\cos(t)) + 3e^{3t}(C_1\cos(t) + C_2\sin(t))$$
$$= e^{3t}[\underbrace{(3C_1 - C_2)}_{K_1}\sin(t) + \underbrace{(3C_1 + C_2)}_{K_2}\cos(t)]$$

Call these K_1 = constants K_2

$$= e^{3t}[K_1\sin(t) + K_2\cos(t)]$$

- Since the cos(t) and sin(t) terms give us an oscillation we can relate $y_h(t)$ and $y_h'(t)$ in a (y, y') plane:



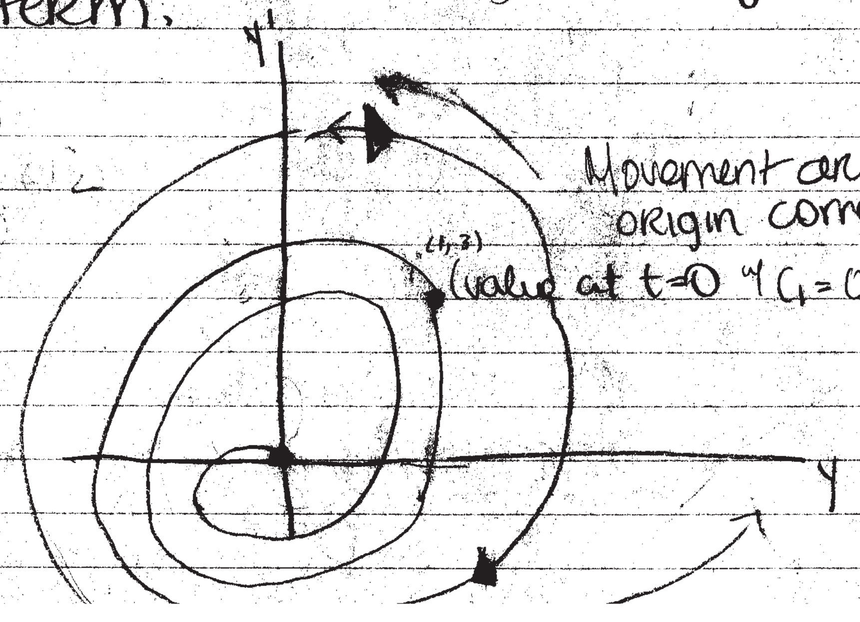
to a complex plane where

these terms in polar coordinates

$(x) - y_h(t)$ is like $e^{3t} r\cos(t)$ and
$(y) - y_h'(t)$ is like $e^{3t} r\sin(t)$

The trajectory of $y_h(t) = e^{3t}(C_1\cos(t) + C_2\sin(t))$ and $y_h'(t) = e^{3t}[K_1\sin(t) + K_2\cos(t)]$ will then take the form of an unstable spiral (due to the oscillation which comes from the cos & sin terms). The magnitude of the spiral will increase and change by the e^{3t} term.

- Point (0,0) = $C_1 = C_2 = 0$
- Values of C_1 & C_2 shift the trajectory in the plane
- Origin is unstable
- called an unstable spiral



Movement around the origin corresponds to oscillation

° From the above graph and our solutions $y_h(t)$ & $y_h'(t)$, we can generalize the case and say:

If α is positive on the $e^{\alpha t}$ term then as $t \to \infty$, the spiral will proceed in the positive direction — away from the origin. (distance from origin $\to \infty$ as $e^{\alpha t} \to \infty$)

If α is negative & $e^{\alpha t}$ is of the form $e^{-\alpha t}$ then as $t \to \infty$, $e^{-\alpha t}$ will $\longrightarrow 0$ and the distance from the origin goes to zero (the spiral will move inward - stable spiral).

In conclusion;

If the roots of the homogeneous equation are complex and the real part is positive then the trajectory on the $(y, v) = (y, y')$ plane goes like positive time & the spiral will go from an initial point (determined by varying C_1 & C_2 values) outward.

• If the real part is negative then the trajectory on the $(y, v) = (y, y')$ plane goes like negative time & the spiral will go from the initial point to the origin (zero) (backward)

★ If the eigenvalues (roots) are of the form $\lambda = \alpha \pm i\beta$

$\alpha > 0 \Rightarrow$ origin is an unstable spiral

$\alpha < 0 \Rightarrow$ " " stable spiral

Problem 4

$y'' + y' - 6y = 0$

characteristic equation: $\lambda^2 + \lambda - 6 = 0$

$(\lambda + 3)(\lambda - 2) = 0$

Two linearly independent solutions:

$y_1(x) = e^{-3x} \quad y_2(x) = e^{2x}$

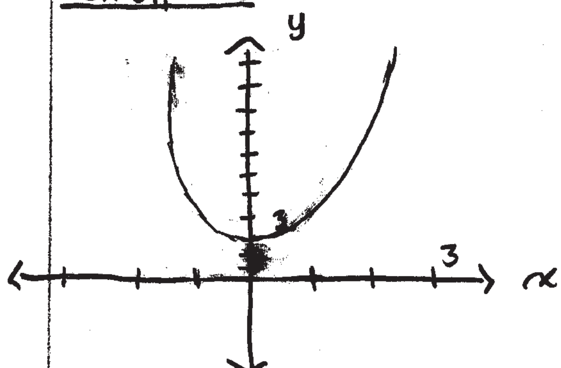
Choosen initial conditions: $c_1 = 1$, $c_2 = 2$

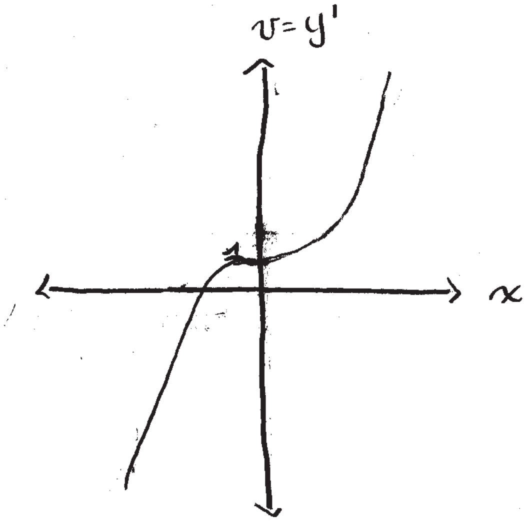
So, $y_h(x) = e^{-3x} + 2e^{2x}$

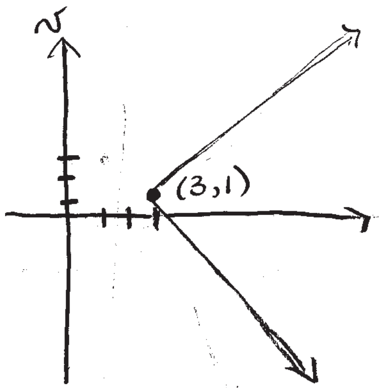
Set $v = y'$

$\therefore \; v = -3e^{-3x} + 4e^{2x}$

Graphs

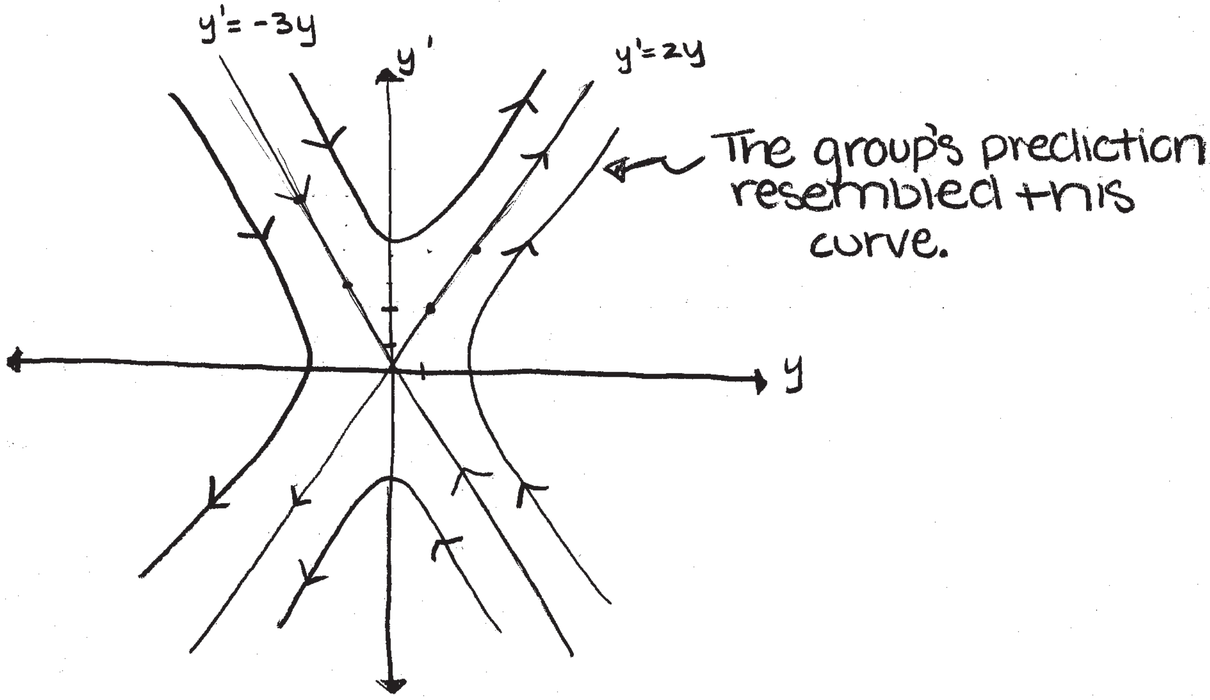






Summary: The $+y'$ term indicates damping in the system, which makes one expect the system to eventually approach zero. The $-6y$ term is the stiffness factor which will always be pushing the system away from zero. Because of the stiffness factor, the system will only go to zero for certain initial conditions (An Example is $c_1 = 1$ & $c_2 = 0$)

The graph of y vs y' looks like:



The origin is unstable, and is called a saddle.