Garrett Vanhoy: Notes on Linear independence

① $\int_0^{2\pi} \sin(mx)\cos(nx)\,dx$ m, n are integers

Integration by parts:

$u = \sin(mx)$ $V = \frac{\sin(nx)}{n}$

$du = m\cos(mx)$ $dV = \cos(nx)$

$$\Rightarrow \int_0^{2\pi} \sin(mx)\cos(nx) = \frac{\sin(mx)\sin(nx)}{n} - \frac{m}{n}\int \sin(nx)\cos(mx)$$

Integration by parts again:

$u = \cos(mx)$ $V = \frac{-\cos(nx)}{n}$

$du = -m\sin(mx)$ $dV = \sin(nx)$

$$\int_0^{2\pi} \sin(mx)\cos(nx) = \frac{\sin(mx)\sin(nx)}{n} - \frac{m}{n}\left[\frac{\cos(mx)(-\cos(nx))}{n} - \frac{m}{n}\int_0^{2\pi}\cos(nx)\sin(mx)\,dx\right]$$

$$\Rightarrow \frac{\sin(mx)\sin(nx)}{n} - \frac{m\cos(mx)\cos(nx)}{n^2} + \frac{m^2}{n^2}\int_0^{2\pi}\cos(nx)\sin(mx)\,dx$$

Since we have $\int_0^{2\pi}\cos(nx)\sin(mx)\,dx$ on both sides we can gather like terms

$$\left(1-\frac{m^2}{n^2}\right)\int_0^{2\pi}\sin(mx)\cos(nx)\,dx = \frac{1}{n}\left(\sin(mx)\sin(nx) - \frac{m}{n}\cos(mx)\cos(nx)\right)\Big|_0^{2\pi}$$

$$= \boxed{\frac{1}{n}\left(\sin(mx)\sin(nx) - \frac{m}{n}\cos(mx)\cos(nx)\right)\Big|_0^{2\pi}}$$

$$= \frac{1}{n}\left(0 - \frac{m}{n}\cos(2\pi m)\cos(2\pi n)\right) - \left(0 - \frac{m}{n}\cos(0)^2\right)$$

$$\frac{1}{n}\left(0-\frac{m}{n}\right)-\left(0-\frac{n}{n}\right)=0$$

2nd Equation   m, n are integers

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$\cos(a-b) = \cos a \cos b + \sin a \sin b$

$\cos(a+b) - \cos(a-b) = 2\sin a \sin b$

$$\int_0^{2\pi} \sin(nx)\sin(mx)\,dx$$

$$\frac{1}{2}\int_0^{2\pi}\left[\cos((n+m)x) - \cos((n-m)x)\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin((n+m)x)}{n+m} - \frac{\sin((n-m)x)}{n-m}\right]_0^{2\pi} \qquad \underline{n \neq m}$$

$$= \frac{1}{2}\left(\frac{\sin((n+m)2\pi)}{n+m} - \frac{\sin((n-m)2\pi)}{n-m}\right) - \frac{1}{2}\left(\frac{\sin((n+m)0)}{n+n} - \frac{\sin((n-m)0)}{n-m}\right)$$

$$= \frac{1}{2}(0-0) - (0-0) = 0$$

If $n = m$ By trig identity

$$\Rightarrow \int_0^{2\pi} \sin(nx)^2\,dx \Rightarrow \frac{1}{2}\int_0^{2\pi}\left(1-\cos\left(\frac{x}{2}\right)\right)dx$$

$$\Rightarrow \frac{1}{2}\left[x - 2\sin(x/2)\right]_0^{2\pi} \Rightarrow \frac{1}{2}\left[\left(2\pi - 2\sin(2\pi/2)\right) - \left(0 - 2\sin\left(\frac{0}{2}\right)\right)\right]$$

$$\Rightarrow \frac{1}{2}(2\pi - 0) - (0-0) = \boxed{\pi}$$

(3rd equation) m, n are integers

$$\int_0^{2\pi} \cos(mx)\cos(nx)\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} \big(\cos((m+n)x) + \cos((m-n)x)\big)\,dx$$

$$= \frac{1}{2}\left[\frac{\sin((m+n)x)}{m+n} + \frac{\sin((m-n)x)}{m-n}\right]_0^{2\pi}$$

$= 0$ however, $m \neq n$ else we divide by zero

If $m=n$ we have

$$\frac{1}{2}\int_0^{2\pi} (\cos(2mx))\,dx + \frac{1}{2}\int_0^{2\pi} \cos(0)\,dx$$

$$= \frac{\sin((m+m)x)}{m+m}\Big|_0^{2\pi} + \frac{1}{2}[x]_0^{2\pi} = \boxed{\pi}$$

Also, $\int_0^{2\pi} \sin(mx)\sin(nx)\,dx$ at $m=n$ $= \boxed{\pi}$

Using the definition of linear independence:

$c_0 \cos(x) + c_1 \cos(2x) + c_2 \sin(x) + c_3 \sin(3x)$

$+ c_4 \sin(5x) + c_5 \cos(5x) = 0$

So, to show that this happens only when all the c's are 0.

If I multiply the left-hand side by $\cos(x)$ and integrate with respect to x:

$$c_0\int_0^{2\pi}(\cos(x))^2dx + c_1\int_0^{2\pi}\cos(x)\cos(2x)dx + c_2\int_0^{2\pi}\cos(x)\sin(x)dx$$

$$+ c_3\int_0^{2\pi}\cos(x)\sin(3x)dx + c_4\int_0^{2\pi}\cos(x)\cos(5x) = 0$$

We see that all integrals are zero except the first.

$$c_0\pi = 0 \Rightarrow c_0 = 0$$

Multiplying the set by each term and integrating will yield the same result and thus the set satisfies the condition for linear independence.