Math 250B (Tuesday Problems Session - 2/10/09)

Problem 1. Last semester, we considered a differential equation called the *logistic equation*, given as

$$\tfrac{dN}{dt} = RN\left(1 - \tfrac{N}{k}\right).$$

This equation has traditionally provided a useful starting point for thinking about population dynamics, where R is called the *per capita growth rate* and k is the *carrying capacity* (both positive constants). Here, we consider the discrete case (in a slightly simplified form) given as

$$x_{n+1} = rx_n\left(1 - x_n\right),$$

such that $x_n \geq 0$ and $r > 1$. This equation is called the *logistic map*, or sometimes (a special case of) the quadratic map. Given an *initial condition* x_1, we can use the map to build a *sequence of iterates*

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_n,\ \ldots$$

The goal of this problem is to study the *properties* of this sequence, as the initial condition x_1 and the parameter r vary.

a. Find any fixed-points $\bar{x}$ of the logistic map, such that $\bar{x} = r\bar{x}\left(1 - \bar{x}\right)$.

b. We now wish to determine the stability of these fixed points. To do this, make a plot of x_{n+1} versus x_n. In order to maintain $x_n \geq 0$ for any value n, state any constraints upon the system (e.g., values r can take).

c. Make a plot for the case $r = 1.5$ and *iterate the map*[1], starting from $x_1 = 0.1$. Record the successive terms in the sequence of iterates and describes what happens to x_n as n gets large. Use this information to specify the stability of the fixed-points.

d. Using the same initial condition as the previous part, use cobweb plots to find what happens when $r = 2, 2.5, 3, 3.2$. (This is hard to do precisely, but try to see if you can get a feel for what is going on). Do you see any new behavior starting to emerge? Describe what you think this indicates in terms of the behavior of the sequence (x_n) as $n \to \infty$. (Hint: make a plot of x_n versus n).

e. At this point, it may be a bit better to explore things numerically. The following web-site contains an applet to examine the behavior of the logistic map for different values of r

[1] This can be done graphically in a simple way. On your plot, add in the line $x_n = x_{n+1}$. Now for the $n = 1$ case, start on the horizontal axis at x_1. Draw a line up to the curve $x_{n+1} = rx_n\left(1 - x_n\right)$ (this tells you what x_2 is). Now make a horizontal connection over to the $x_n = x_{n+1}$ line (giving you your new horizontal location for determining the next iteration). From here make the vertical connection back to the $x_{n+1} = rx_n\left(1 - x_n\right)$ curve (this tells you what x_3 is). Repeat..... (these are sometimes called *cobweb plots*).

(http://www.cmp.caltech.edu/ mcc/Chaos_Course/Demonstrations.html). What happens as you start to increase r beyond 3.2?

f. Beyond $r = 3.57$, something new happens: *chaos*! Pick a value of r greater than 3.57. Now try changing your initial condition (x_1) slightly. What happens? How does this differ from smaller values of r?

Summary
Via the logistic map, you just wandered into the realm of chaos. Even though the behavior of a chaotic system may appear totally random, it is often entirely deterministic (as is the case with the logistic map). In this exploration, you looked at the sequence of iterates of the logistic map and described its asymptotic behavior (from constant, to periodic, and eventually aperiodic), based upon the value in r. As we discussed last semester, this major change in the system's behavior due to a parameter varying is an example of a *bifurcation*. Furthermore, the sensitivity to the initial conditions (for larger values of r in the case of the logistic map) is a trademark of chaotic systems.

Problem 2. This problem examines a very famous example of what is called a *second order difference equation* (as x_{n+1} depends not only on x_n but x_{n-1} as well). It stems from the mathematician Leonardo of Pisa (also called Fibonacci) who posed the following problem regarding (idealized) rabbit procreation in the early 13'th century: You start with a pair of newborn rabbits. Over the season, the pair reproduces another pair. Over the next season, both pairs reproduce (another pair each) and the original stop reproducing (i.e. they only reproduce for two seasons). All offspring follow the same pattern. How many pairs of newborn rabbits are subsequently produced for a given season?
The answer to this question is given by the following equation (known as the *Fibonacci recursion*),

$$s_{n+1} = s_n + s_{n-1}, \qquad n \geq 2,$$

where s_n is the number of newborn rabbits in the n'th season and for which s_1 and s_2 are known (0 and 1 respectively in this example). The value s_n is known as a Fibonacci number.

a. Draw a picture in order to determine the first seven Fibonacci numbers. Do you think the Fibonacci sequence converges? Why or why not?

b. Consider the sequence (u_n) defined by $u_n = s_{n+1}/s_n$, where (s_n) is the Fibonacci sequence. Does the sequence (u_n) converge? If so, what does it converge to?

c. Divide both sides of the Fibonacci recursion by s_n and assume that

$$\lim_{n\to\infty} \frac{s_{n+1}}{s_n} = \lambda$$

Show then that this leads to the quadratic equation in the limit for large n:

$$\lambda^2 - \lambda - 1 = 0,$$

and show that its roots have opposite signs.

d. Call φ the positive root of this equation (it is called the *golden ratio*[2]), and find an expression for φ. Also express the other root in terms of φ.

e. Show that if α and β are two real numbers, then the sequence (v_n) defined by

$$v_n = \alpha\varphi^n + \beta(1-\varphi)^n$$

satisfies the Fibonacci recursion defined above.

f. Find values for α and β such that the first two terms in the sequence are 0 and 1. Is the resulting sequence the same as the *Fibonacci sequence*?

g. As alluded to re the Greeks, Fibonacci numbers appear in an astounding number of places in nature. Look at the picture below of a Helianthus flower. Notice the spiral patterns emanating from the center. One can visualize spirals rotating outwards in both a clockwise and counter-clockwise directions. Count the total number of spirals going outwards in both directions. Do you see an connection to the Fibonacci numbers?

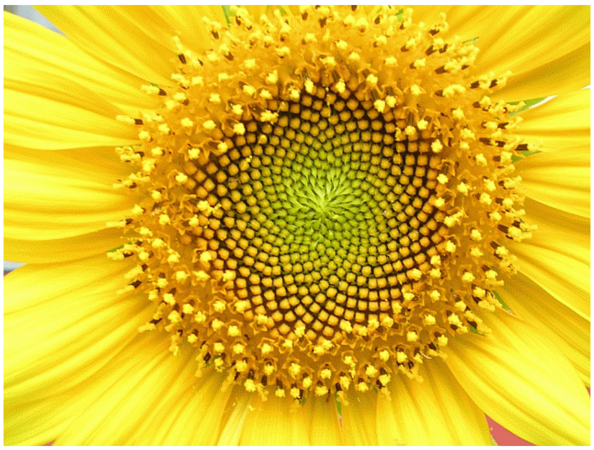

Figure 1: A Helianthus, subfamily of flowers in the plant kingdom that are known for their unusual height. This particular picture (from http://en.wikipedia.org/wiki/File:Helianthus_whorl.jpg) shows a sunflower.

[2]The greeks considered a rectangle whose sides had this ratio to be the most aesthetically pleasing proportion a rectangle can have and incorporated it into many facets of life such as their architecture!

h. In a similar way, count the number of petals in the flowers shown in the photographs below. What do you notice?

Figure 2: Various flowers.