Math 250B (Spring 2009) - Exam I **Name:**

Instructions:

- Show all work clearly in order to get full credit. Points can be taken off if it is not clear to see how you arrived at your answer (even if the final answer is correct).
- When you do use your calculator, sketch all relevant graphs and explain all relevant mathematics.
- Circle your final answers.
- Please keep your written answers brief; be clear and to the point.
- This test has 6 problems and is worth 100 points, plus extra credit at the end. It is your responsibility to make sure that you have done all the problems!

1. (*12 points*) Does the following integral converge or diverge? Explain your reasoning. If it converges, give its value correct to three decimal places, using the method of your choice.

$$\int_0^1 \frac{dx}{\sqrt{1-x}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x}} = \lim_{b \to 1^-} \int_0^b \frac{dx}{\sqrt{1-x}} = \lim_{b \to 1^-} \left[-2\sqrt{1-x}\right]_0^b$$

$$= \lim_{b \to 1^-} \left[-2\sqrt{1-b} + 2\right] = 2$$

So the integral converges to $\boxed{2}$ (exactly).

2. (*20 points*) Does the following integral converge or diverge? Give a complete justification of your answer.

$$\int_0^\infty (2+\cos(t))\, e^{-t}\, dt.$$

<u>Note</u>: You can evaluate this integral, using integration by parts twice. However, the question was to decide whether the integral converged or diverged, so if you can answer the question without integrating, it will save you some time.

Since $-1 \leq \cos(t) \leq 1$, $1 \leq 2+\cos(t) \leq 3$ and

$$e^{-t} \leq (2+\cos(t))\, e^{-t} \leq 3\, e^{-t}$$

So $$0 \leq \int_0^\infty (2+\cos(t))\, e^{-t}\, dt \leq \int_0^\infty 3\, e^{-t}\, dt$$

Now, $$\int_0^\infty 3\, e^{-t}\, dt = \lim_{b\to\infty} \left[3\, e^{-t}\right]_0^b = \lim_{b\to\infty} \left[-3\, e^{-t}\right]_0^b$$

$$= \lim_{b\to\infty} (-3\, e^{-b} + 3) = 3 \qquad \text{since } \lim_{b\to\infty} e^{-b} = 0.$$

Therefore $$0 \leq \int_0^\infty (2+\cos(t))\, e^{-t}\, dt \leq 3$$

So the integral <u>converges</u>.

3. (*20 points*) Determine the length of the following curve from $x = 1$ to $x = 6$:

$$h(x) = \frac{x^4}{4} + \frac{1}{8x^2}$$

$$\ell = \int_1^6 \sqrt{1 + [h'(x)]^2}\, dx$$ using the formula for the arc length.

$$h'(x) = x^3 - \frac{2}{8x^3} = x^3 - \frac{1}{4x^3}$$

$$\ell = \int_1^6 \sqrt{1 + \left(x^3 - \frac{1}{4x^3}\right)^2}\, dx = \int_1^6 \left(1 + x^6 - \frac{2}{4} + \frac{1}{16x^6}\right)^{1/2} dx$$

$$= \int_1^6 \left(\frac{1}{2} + x^6 - \frac{1}{16x^6}\right)^{1/2} dx = \int_1^6 \sqrt{\left(x^3 + \frac{1}{4x^3}\right)^2}\, dx$$

$$= \int_1^6 \left(x^3 + \frac{1}{4x^2}\right) dx = \left[\frac{x^4}{4} - \frac{1}{8x^2}\right]_1^6 = \frac{36^2}{4} - \frac{1}{8 \cdot 36} - \frac{1}{4} + \frac{1}{8}$$

$$= 9 \cdot 36 - \frac{37}{8 \cdot 36} = \frac{23\,275}{288} \simeq \boxed{323.87}$$ units of length.

4. (*10 points*) **(a)** Find the length of the curve described parametrically as

$$x = 2\sin(t), \qquad y = 2\cos(t), \qquad 0 \le t \le 2\pi.$$

$$\ell = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \int_0^{2\pi} \sqrt{(2\cos(t))^2 + (-2\sin(t))^2}\, dt$$

$$= \int_0^{2\pi} \sqrt{4(\cos^2(t) + \sin^2(t))}\, dt = \int_0^{2\pi} 2\, dt = 4\pi \text{ units of length}$$

So $\boxed{\ell = 4\pi \text{ units of length}}$.

(b) Can you describe some other means to determine what the arc length of this curve is?

The curve $\begin{cases} x = 2\sin(t) \\ y = 2\cos(t) \end{cases}$ for $0 \le t \le 2\pi$ corresponds

to a circle of radius 2 in polar coordinates.

Its length is therefore $2\pi r = 2\pi \cdot 2 = 4\pi$.

5. (*13 points*) State whether the following integral converges or not. If it converges, state what it converges to.

$$\int_3^\infty \frac{\pi e^4}{(t+4)^3}\,dt$$

$$\int_3^\infty \frac{\pi e^4}{(t+4)^3}\,dt = \lim_{b\to+\infty} \int_3^b \frac{\pi e^4}{(t+4)^3}\,dt = \lim_{b\to+\infty} \left[\frac{\pi e^4}{(t+4)^2}\left(-\frac{1}{2}\right)\right]_3^b$$

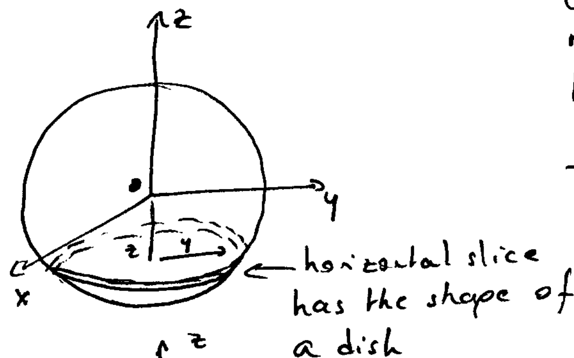
$$= \lim_{b\to+\infty}\left(-\frac{\pi e^4}{2(b+4)^2} + \frac{\pi e^4}{2(7)^2}\right) = \frac{\pi e^4}{2\cdot 49} = \frac{\pi e^4}{98}$$

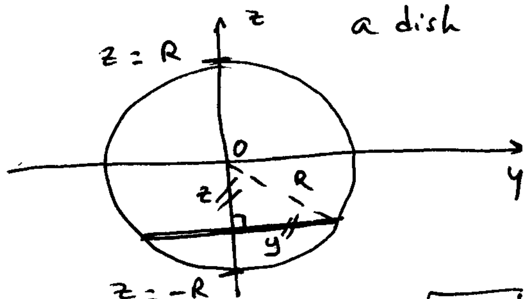
$\boxed{\text{So the integral \underline{converges} to } \frac{\pi e^4}{98}}.$

6. (*25 points*) The goal of this problem is to set up an integral to calculate the volume V of a sphere of radius R, using the method (slicing, shells, solids of revolution, etc) of your choice. Then, to evaluate this integral.

(a) Draw a picture to set up the integral for V.

You can either use horizontal (or vertical slices), or use the shell method (by revolving a $\frac{1}{2}$-disk about the vertical axis). Here, we use horizontal slices.





$y^2 + z^2 = R^2 \Rightarrow y = \sqrt{R^2 - z^2}$

- Area of each slice $= \pi\, y^2 = \pi (R^2 - z^2)$
- Volume of each slice $= \pi (R^2 - z^2)\, \Delta z$
- Volume of sphere

$$\lim_{\Delta z \to 0} \sum_{i=1}^{n} \pi (R^2 - z_i^2)\, \Delta z = \boxed{\pi \int_{-R}^{R} (R^2 - z^2)\, dz = V}$$

(b) Evaluate the integral you found in part (a) to determine the volume of the sphere.

So $$V = \pi \int_{-R}^{R} (R^2 - z^2)\, dz = \left[\pi \left(R^2 z - \frac{z^3}{3} \right) \right]_{-R}^{R}$$

$$= \pi \left(R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3} \right) = 2\pi R^3 \left(1 - \frac{1}{3}\right) = 2\pi R^3 \frac{2}{3}$$

i.e. $$\boxed{V = \frac{4}{3} \pi R^3.}$$

Extra Credit (10 points)
Show that if

$$g(\theta) = \frac{e^{\theta} + e^{-\theta}}{2}$$

then the length of the curve $g(\theta)$ between $\theta = 0$ and $\theta = \alpha$ (where $\alpha > 0$) is given by $\frac{dg}{d\theta}(\alpha)$.

Note that $g(\theta) = \cosh(\theta)$. Thus, $g'(\theta) = \sinh(\theta)$ and

length of curve $= \ell = \int_0^{\alpha} \sqrt{1 + [g'(\theta)]^2}\, d\theta$

$$\ell = \int_0^{\alpha} \sqrt{1 + \sinh^2(\theta)}\, d\theta$$

$$= \int_0^{\alpha} \sqrt{\cosh^2(\theta)}\, d\theta \qquad \text{since } \cosh^2(\theta) - \sinh^2(\theta) = 1$$

$$= \int_0^{\alpha} \cosh(\theta)\, d\theta \qquad \text{since } \cosh(\theta) > 0 \text{ for all } \theta$$

$$= \Big[\sinh(\theta)\Big]_0^{\alpha} \qquad \text{since } \frac{d}{d\theta}\sinh(\theta) = \cosh(\theta)$$

$$= \sinh(\alpha) - \sinh(0) = \sinh(\alpha) \qquad \text{since } \sinh(0) = 0$$

$$= \frac{d}{d\alpha}\left[\cosh(\alpha)\right] \qquad \text{since } \frac{d}{d\alpha}\cosh(\alpha) = \sinh(\alpha)$$

$$= g'(\alpha).$$