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Generalized and attenuated Radon transforms: restorative approach to the numerical inversion

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Abstract. The problem of the function reconstruction on its line integrals with known weight function is considered. The approach studied consists of treating the attenuated projections by the Radon transform inversion formula and considering the result of the inversion as a distorted image. A helpful formula describing the distortion is obtained. The norm of the distortion operator is estimated and several iterative restoration algorithms based on the integral transfers are investigated. The results of the numerical inversion of the attenuated Radon transform are presented to demonstrate the features of the algorithms.

1. Introduction

Many kinds of computerized tomography are connected with the reconstruction of an unknown function from its line integrals. The mathematics of this problem may frequently be described in terms of the integral Radon transform. In such cases various reconstruction algorithms have been designed on the basis of either the known Radon transform inversion formula or some other approaches [1]. The situation is quite different in emission tomography where the measured projections due to an attenuation are line integrals with known weight function depending on the angular variable. If the radiation attenuation in the medium is negligible then the weight function is close to unity and the technique of Radon transform inversion is applicable. In the opposite case the problem of function reconstruction on its line integrals with known weight function requires a specific consideration.

Extensive results have been obtained for solving the inversion problem for the so-called exponential Radon transform with uniform attenuation [2, 3]. However, the constant attenuation approximation is frequently not applicable. In more general cases algorithms based on the iterative correction of the attentuation effects [4–6] are applied to solve this problem. A theory of these algorithms is rather heuristical and needs in further analysis.

In this paper we investigate a two-stage approach for the numerical inversion both of attenuated and generalized Radon transforms. In the first stage attenuated projections are treated by the Radon transform inversion formula. In the second stage the result of the inversion is considered as a distorted image and the problem of the original image restoration is solved. Such an approach is typical for iterative algorithms with attenuation correction [4, 6, 7].

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To analyse the problem in detail we obtain a helpful formula describing the distortion. Then we investigate an equation connecting the original and distorted images and estimate the norm of the distortion operator. This allows us to derive the known Chang's algorithm [4] and some other algorithms as a Neumann series for the above equation and to evaluate the convergence domain of the algorithms. We perform the numerical inversion of the attenuated Radon transform to demonstrate the features of the developed methods.

2. Formulation of the problem

Measured projections $I(p, \omega)$ are related to the medium emission coefficient $\epsilon(x)$ by the generalized Radon transform [8]:

$$I(p,\omega) = \int_{x \cdot \omega = p} \epsilon(x) W(x,\omega)\, \mathrm{d}x \equiv R_W \epsilon(x) \tag{1}$$

where R_W is the generalized Radon transform operator, $x \in \mathbb{R}^2$, $\omega \in S^1$, $p \in \mathbb{R}^1$. If $W(x,\omega) \equiv 1$ then (1) is reduced to the usual Radon transform, the opertor of which is denoted simply by R.

In many emission tomography problems a weight function $W(x,\omega)$ is related to the medium attenuation coefficient $\mu(x)$ by the following dependence:

$$W(x,\omega) = \exp\left(\int_0^\infty \mu(x + t\omega^{\mathrm{T}})\, \mathrm{d}t\right) \tag{2}$$

where $\omega^{\mathrm{T}} \in S^1$, $\omega^{\mathrm{T}} = (\cos\varphi, \sin\varphi)$, $\omega = (-\sin\varphi, \cos\varphi)$. The transform (1) with the weight function (2) is called an attenuated Radon transform. In the case of the constant attenuation coefficient ($\mu(x) \equiv \mu_0$) and a convex support of an attenuation, the exact inversion formula exists for this transform [2, 3]. However, the constant attenuation approximation is often not applicable. Moreover, the dependence $W(x,\omega)$ on the integral $\int_0^\infty \mu(x + t\omega^{\mathrm{T}})\, \mathrm{d}t$ is not exponential in some problems. This appears, for example, when the intensity of the infrared radiation of molecular gases is measured within a wide frequency range [9]. That is why we consider the problem in the more general form (1) without the concretization of the weight function form.

We consider a result of the Radon inversion of the attenuated projections $\epsilon_D(x)$ as a distorted image:

$$\epsilon_D(x) \equiv R^{-1} I(p,\varphi) = R^{-1} R_W \epsilon(x) = \epsilon(x) - D\epsilon(x) \tag{3}$$

where $\epsilon(x)$ is original image, the distortion operator D is defined by

$$D\epsilon(x) \equiv R^{-1}(R - R_W)\epsilon(x) = R^{-1}\left(\int_{x \cdot \omega = p} \epsilon(x)(1 - W(x,\omega))\, \mathrm{d}x\right) \tag{4}$$

and the Radon transform inversion operator is defined by

$$R^{-1} I(p,\omega) = \int_{S^1} \mathrm{d}\omega \int_{-\infty}^{+\infty} \hat{I}(\rho,\omega)\hat{\eta}(\rho) \mathrm{e}^{2\pi i \rho (x \cdot \omega)}\, \mathrm{d}\rho$$

where [8]

$$\hat{I}(p,\varphi) = \int_{-\infty}^{+\infty} I(p,\omega)\mathrm{e}^{-2\pi i \rho p}\,\mathrm{d}p.$$

The Fourier transformation of the filter function $\eta(p)$ is equal to:

$$\hat{\eta}(\rho) = (\tfrac{1}{2})|\rho|.$$

In the next section we obtain an expression for the distortion $D\epsilon(x)$ under the assumption that $\epsilon(x)$ is a finite function with the support $\bar{\Omega}$, $\epsilon(x) \in L^2(\bar{\Omega})$ and $0 < W(x,\omega) < 1$ on $\bar{\Omega} \times [0, 2\pi]$.

3. An expression for the distortion

The weight function $(1 - W(x,\omega))$ in the distortion definition (4) may be expanded as a Fourier series in the angular variable φ as follows:

$$1 - W(x,\omega(\varphi)) = W_0(x) + \sum_{k=1}^{\infty} [W_k(x)\mathrm{e}^{ik\varphi} + W_{-k}(x)\mathrm{e}^{-ik\varphi}] \tag{5}$$

where

$$W_k(x) = \frac{1}{2\pi}\int_0^{2\pi} (1 - W(x,\omega(\varphi))\mathrm{e}^{-ik\varphi}\,\mathrm{d}\varphi \qquad k \in \mathbb{Z}.$$

Note that $|W_k(x)| < 1$ for any k and x; $W_{-k}(x) = \bar{W}_k(x)$ for $\forall k \in \mathbb{Z}$ and $W_0(x)$ is a bounded real function: $0 < W_0(x) \leqslant 1$. Therefore, the distortion may be present in the form:

$$D\epsilon(x) = R^{-1}\left(R(\epsilon(x)W_0(x)) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathrm{e}^{ik\varphi} R[\epsilon(x)W_k(x)]\right)$$

$$= W_0(x)\,\epsilon(x) + D_1\epsilon(x)$$

where

$$D_1\epsilon(x) \equiv R^1\left(\sum_{k\in\mathbb{Z}\setminus\{0\}} [\mathrm{e}^{ik\varphi} R(\epsilon(x)W_k(x))]\right). \tag{6}$$

Owing to the linearity of the transforms R and R^{-1} every addend of the sum (6) may be present as a convolution:

$$R^{-1}(\mathrm{e}^{ik\varphi} R(\epsilon(x)W_k(x))) = [W_k(x)\epsilon(x)] ** d_k(x)$$

where $d_k(x) \equiv R^{-1}[\mathrm{e}^{ik\varphi} R\sigma(x)]$, $\sigma(x)$ is 2D Dirac delta-function and $**$ denotes 2D convolution. Because of $R\sigma(x) = \sigma(p)$ [8] the expression for the $d_k(x)$ may be simplified. In the polar coordinates (r, θ) it is written as:

$$d_k(r,\theta) = R^{-1}[\mathrm{e}^{ik\varphi}\sigma(p)]$$

$$= \int_0^{2\pi} \mathrm{e}^{ik\varphi} \int_{-\infty}^{+\infty} \frac{|\rho|}{2}\,\mathrm{e}^{2\pi i \rho r \sin(\theta-\varphi)}\,\mathrm{d}\rho\,\mathrm{d}\varphi$$

$$= e^{ik\theta} \int_{-\infty}^{+\infty} \frac{|\rho|}{2} \int_0^{2\pi} e^{-2\pi i\rho r \sin\varphi + i\kappa\varphi} \, d\varphi \, d\rho$$

$$= 2\pi e^{ik\theta} \int_{-\infty}^{+\infty} \frac{|\rho|}{2} J_k(2\pi\rho r) \, d\rho$$

$$= \begin{cases} |k| \dfrac{e^{ik\theta}}{r^2} & k \text{ even}, k = 0 \\ 0 & k \text{ odd}. \end{cases} \tag{7}$$

Thus, the distortion is related to the original image by

$$\begin{aligned} D\epsilon(x) &= W_0(x)\epsilon(x) + D_1\epsilon(x) \\ D_1\epsilon(x) &= \sum_{k\in\mathbb{Z}\setminus\{0\}} [\epsilon(x)W_{2k}(x)] ** d_{2k}(x) \end{aligned} \tag{8}$$

where the $d_{2k}(x)$ are described by (7). Note that the Fourier transformation of $d_{2k}(r, \theta)$ is equal to:

$$F_2(d_{2k}(r,\theta)) = e^{2ki(\alpha+\pi/2)}$$

where F_2 is the 2D Fourier transform operator, and α is the angular variable in Fourier space. Therefore (8) may be expressed as an operator series of products and Fourier transforms:

$$D_1\epsilon(x) = F_2^{-1} \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{2ki(\alpha+\pi/2)} F_2[\epsilon(x)W_{2k}(x)] \tag{9}$$

where F_2^{-1} is the inverse 2D Fourier transform operator.

4. A norm of the distortion operator

Because a Fourier transform does not change the L_2-norm of the function, the following estimate may be obtained for the L_2-norm of the distortion in accordance with (8) and (9):

$$\|D\epsilon(x)\| \leqslant \sum_{k\in\mathbb{Z}} \|\epsilon(x)W_{2k}(x)\| \leqslant \|\epsilon(x)\| \sum_{k\in\mathbb{Z}} \max_{\Omega} |W_{2k}(x)| \tag{10}$$

where the function L_2-norm is defined by:

$$\|f(x)\|^2 = \int_\Omega f^2(x) \, dx.$$

If the series in (10) converges then the distortion operator is limited in $L_2(\bar{\Omega})$:

$$\|D\| = \sup \frac{\|D\epsilon(x)\|}{\|\epsilon(x)\|} \leqslant M = \sum_{k\in\mathbb{Z}} \max_{\bar{\Omega}} |W_{2k}(x)|. \tag{11}$$

The sufficient condition for the convergence of (10) is the existence of the limited in $L_2(\Omega)$ weight function second derivative on φ.

Lemma 1. If $\exists A > 0$, such that

$$\int_0^{2\pi} \left(\frac{\partial^2}{\partial \varphi^2} (1 - W(x, \omega(\varphi)) \right)^2 \mathrm{d}\varphi \leqslant A^2 \qquad \forall x \in \bar{\Omega}$$

then the series $\Sigma_{k \in \mathbb{Z} \max_{\bar{\Omega}} |W_{2k}(x)|}$ converges and the following estimate is valid: $M \leqslant \max_{\bar{\Omega}} |W_0(x)| + \frac{1}{3} A \pi^2$.

Proof. It follows from the lemma condition that, $\forall x \in \bar{\Omega}$,

$$\sum_{k \in \mathbb{Z} \setminus \{0\}} k^4 |W_k(x)|^2 = \int_0^{2\pi} \left(\frac{\partial^2}{\partial \varphi^2} (1 - W(x, \omega(\varphi)) \right)^2 \mathrm{d}\varphi \leqslant A^2.$$

Therefore, $\forall x \in \bar{\Omega}$ and $\forall k \in \mathbb{Z} \setminus \{0\} | W_k(x)| \leqslant A/k^2$ and

$$\sum_{k \in \mathbb{Z}} \max_{\bar{\Omega}} |W_k(x)| \leqslant \max_{\bar{\Omega}} |W_0(x)| + \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{A}{2k^2} = \max_{\bar{\Omega}} |W_0(x)| + \tfrac{1}{12} A \pi^2.$$

The proof is completed.

Equation (3) may be considered as an integral Fredholm equation of the second kind. If $M < 1$, then $\|\epsilon_D(x)\| < 1$ and this equation has the unique solution which may be presented as a Neumann series:

$$\epsilon(x) = \sum_{j=0}^{\infty} D^j \epsilon_D(x). \tag{12}$$

Therefore, in this case the inverse generalized Radon transform operator R_W^{-1} exists and it may be defined as a resolvent of the kernel D:

$$R_W^{-1} = \sum_{j=0}^{\infty} D^j.$$

The L_2-norm of the inverse operator is also limited in this case:

$$\| R_W^{-1} \| \leqslant \frac{1}{1 - M}.$$

This allows us to derive several iterative algorithms available for the numerical restoration of the original image and, in this way, for the numerical inversion of the generalized Radon transform.

5. Restoration algorithms

We assume in this section that $M = 1 - c$, where c is some positive constant less than unity.

The numerical calculation of the series (12) on the basis of iteration using of direct and inverse Radon transforms is a simple restoration algorithm, i.e.

$$\epsilon^{(k+1)}(x) = P\epsilon_D(x) + PR^{-1}(R - R_W)\epsilon^{(k)}(x) \tag{13}$$

where P is an operator that stands for multiplying on a characteristic function of Ω. An arbitrary function limited in $L_2(\bar{\Omega})$ may be chosen to be an initial approximation $\epsilon(x)$;

the convergence rate in the sense of $L_2(\bar{\Omega})$ is the same or faster than that for the geometric progression with the index M. The calculation of the inverse Radon transform in (13) as well as the calculation of $\epsilon_D(x)$ may be fulfilled either by using a filtered back-projection algorithm or by direct Fourier reconstruction [1, 8].

A more efficient restoration algorithm may be obtained as a Neumann series for equation (3) transformed using (8):

$$\epsilon(x) - \frac{D_i\epsilon(x)}{1 - W_0(x)} = \frac{\epsilon_D(x)}{1 - W_0(x)}.$$

The next approximation $\epsilon^{(k+1)}(x)$ is calculated as:

$$\epsilon^{(k+1)}(x) = \frac{P\epsilon_D(x)}{1 - w_0(x)} + \frac{PD_1\epsilon^{(k)}(x)}{1 - W_0(x)} \tag{14}$$

where the numerical realization of the operator D_1 is carried out by

$$D_l\epsilon^{(k)}(x) = R^{-1}(R - R_W)\epsilon^{(k)}(x) - W_0(x)\epsilon^{(k)}(x).$$

Actually, this algorithm coincides with Chang's corrected method [4] generalized in the case of arbitrary weight function and attenuation coefficient. The convergence rate of the algorithm is defined by the L_2-norm of the corresponding operator:

$$\left\| \frac{D_i}{1 - W_0(x)} \right\| \leqslant m = \frac{M_1}{1 - M_0} \tag{18}$$

where

$$\|D_1\| \leqslant M_1 = \sum_{k \in \mathbb{Z}\setminus\{0\}} \max_{\bar{\Omega}} |W_{2k}(x)|$$

$$M_0 = \max_{\bar{\Omega}} |W_0(x)|.$$

Let us show that $m < M$, i.e. the obtained estimate of the convergence rate for the algorithm (14) is better than that for (13).

Lemma 2. If $M = 1 - c$, where $0 < c < 1$, then $m < M$.

Proof. Note that $M_0 + M_1 = M$. Then

$$1 - M_0 - M_1 - c = 0.$$

We obtain, by multiplying of both parts of the equation by M_0 and by adding M_1,

$$M_0 - M_0^2 - M_1M_0 + M_1 - cM_0 = M_1.$$

Dividing by $(1 - M_0)$ gives:

$$M_1 + M_2 - \frac{cM_0}{1 - M_0} = \frac{M_1}{1 - M_0} = m$$

and therefore

$$m = M - \frac{cM_0}{1 - M_0}.$$

The proof is completed.

If M is equal to unity, then m is equal to unity too. This means that the obtained estimate for the convergence range of the algorithm (14) is same as that for (13).

If we truncate the series (9) then algorithms analogous to both (13) or (14) may be designed on the basis of the 2D FFT. Namely, the operator D may be presented as a sum

$$D\epsilon(x) = D_T\epsilon(x) + D_R\epsilon(x)$$

where

$$D_T\epsilon(x) \equiv W_0(x)\epsilon(x) + D_{1T}\epsilon(x)$$

$$D_{1T}\epsilon(x) \equiv F_2^{-1} \sum_{k\in[-N,N]\setminus\{0\}} e^{2ki(\alpha+\pi/2)} F_2[\epsilon(x)W_{2k}(x)]$$

$$D_R\epsilon(x) \equiv F_2^{-1} \sum_{k\in\mathbb{Z}\setminus[-N,N]} e^{2ki(\alpha+\pi/2)} F_2[\epsilon(x)W_{2k}(x)] \tag{15}$$

and iterations (13) and (14) are fulfilled by using the operators D_T and D_{1T} respectively:

$$\epsilon^{(k+1)}(x) = P\epsilon_D(x) + PD_T\epsilon^{(k)}(x) \tag{16}$$

$$\epsilon^{(k+1)}(x) = \frac{P\epsilon_D(x)}{1 - W_0(x)} + \frac{PD_{1T}\epsilon^{(k)}(x)}{1 - W_0(x)}. \tag{17}$$

Both algorithms (16) and (17) converge to the approximate image $\epsilon_A(x)$

$$\epsilon_A(x) = \epsilon(x) - \epsilon_E(x).$$

The error $\epsilon(x)$ may be estimated in $L_2(\bar{\Omega})$ by:

$$\|\epsilon_E(x)\| \leqslant \frac{1}{1-M}\|D_R\|\,\|\epsilon(x)\| \leqslant \frac{1}{1-M}\|\epsilon(x)\| \sum_{k\in\mathbb{Z}\setminus[-N,N]} \max_{\bar{\Omega}} |W_{2k}(x)|.$$

Since $W_{-2k}(x)\epsilon(x) = \overline{W_{2k}(x)\epsilon(x)}$, fulfilling the 2D Fourier transform $N+1$ times allows us to calculate $D_{1T}\epsilon(x)$ according to (15). This may be carried out using the 2D FFT. Hence, a fast restoration may be performed with (16) or (17) if the weight function dependence on the angular variable is described by the Fourier series with a few harmonics.

6. Application to the attenuated Radon transform: numerical results

The above theory and algorithms may be applied to the numerical inversion of the attenuated Radon transform. For example, if an attenuation coefficient $\mu(x)$ is double differentiable and vanishes to zero with the first and second derivatives on the bound of the support Ω, then the exponential weight function (2) satisfies the conditions of lemma 1. In the case when the attenuation coefficient and its derivatives have first-kind discontinuities on the support bound, the problem may be solved in some cases by expanding the support and by redefining the attenuation coefficient to eliminate discontinuities. An appropriate correction of the projections is required in this case.

The simulation of the numerical inversion of the attenuated Radon transform (with exponential weight function) have been performed to demonstrate the features of the

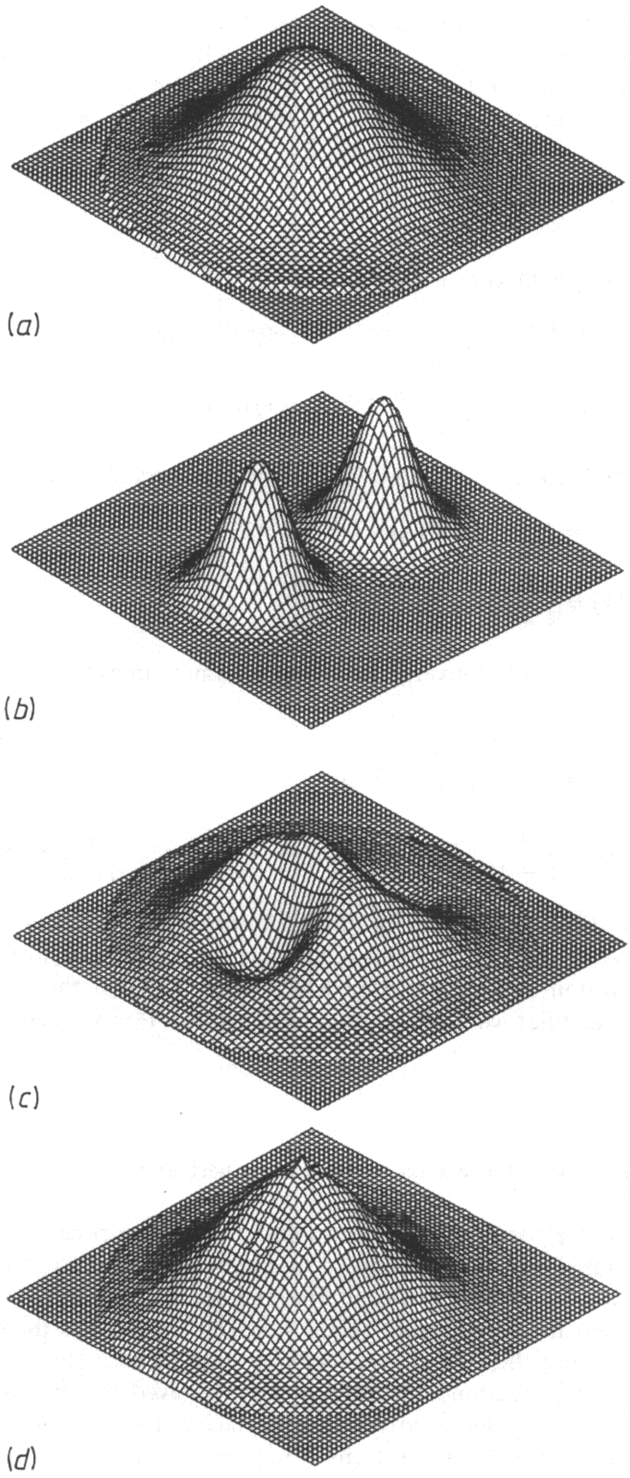

Figure 1. (*a*) original image $\epsilon(x)$; (*b*) attenuation coefficient $\mu(x)$; (*c*) distorted image $\epsilon_D(x)$; (*d*) third iterate of the algorithm (17).

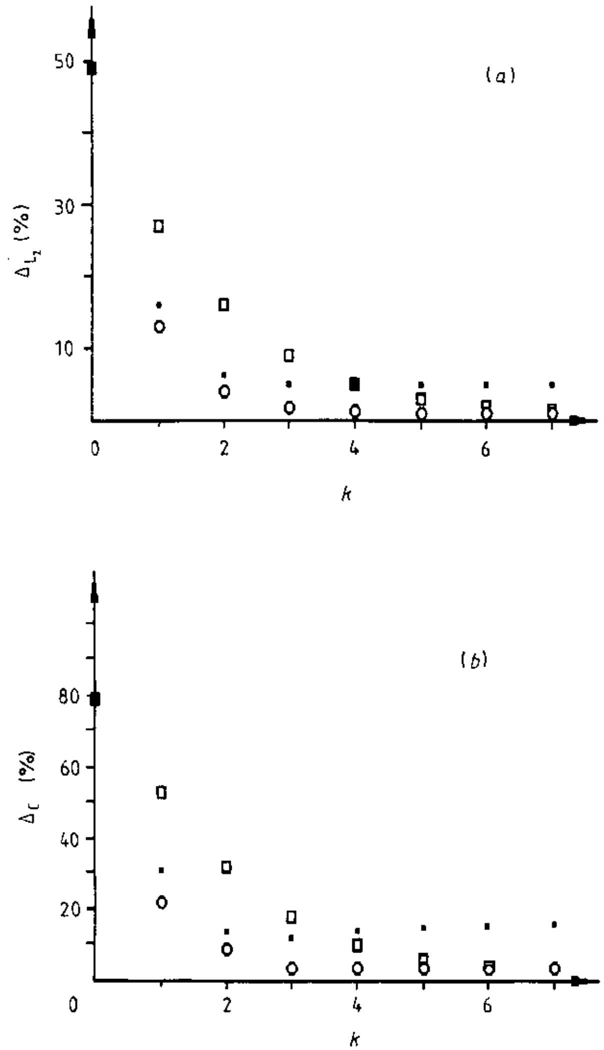

Figure 2. Error values versus the iterate number k for (*a*) L_2-norm and (*b*) C-norm for: algorithm (13) (□); algorithm (14) (○); algorithm (17) (■).

above algorithms. The emission coefficient $\epsilon(x)$ (original image) and the attenuation coefficient $\mu(x)$ were defined on the unit circle by

$$\epsilon(x) = \exp\left[-\frac{(x_1 - 0.1)^2 + x_2^2}{2 \times 0.35^2}\right]$$

$$\mu(x) = 4\exp\left[-\frac{(x_1 + 0.4)^2 + x_2^2}{2 \times 0.15^2}\right] + 4\exp\left[-\frac{(x_1 - 0.4)^2 + x_2^2}{2 \times 0.15^2}\right]$$

where $x = (x_1, x_2)$. These functions are shown in figures 1(*a*) and 1(*b*) respectively. Eleven equidistant projections have been calculated in the angular variable range of $[0, 2\pi]$. We evaluated the reconstruction quality on the values of the error in L_2-

and C-norms, i.e.

$$\Delta_{L_2} = \left[\frac{\int_\Omega (\epsilon^{(k)}(x) - \epsilon(x))^2 \, dx}{\int_\Omega \epsilon(x)^2 \, dx}\right]^{1/2}$$

$$\Delta_C = \frac{\max_{\bar{\Omega}} |\epsilon^{(k)}(x) - \epsilon(x)|}{\max_{\bar{\Omega}} |\epsilon(x)|}$$

An algorithm based on the direct application of the Radon inversion formula with spline regularization was applied to calculate the inverse Radon transform R^{-1}. The distorted image $\epsilon_D(x)$ is shown in figure 1(c) (compare with figure 1(a)). Values of the error $\Delta_{L_2}^2$ and Δ_C for $\epsilon_D(x)$ are equal to 49% and 79% correspondingly.

Parameters M and m were evaluated approximately on the four first harmonics:

$$M \approx \sum_{k \in [-3,3]} \max_{\bar{\Omega}} |W_{2k}(x)| \approx 1.36$$

$$m \approx \frac{1}{1 - M_0} \sum_{k \in [-3,3] \setminus \{0\}} \max_{\bar{\Omega}} |W_{2k}(x)| \approx 1.83$$

In spite of parameters M and m being outside the domain of proved convergence, the examined algorithms have converged rather quickly. Algorithms (14) and (17) have reached convergence on the third iterate. Algorithm (13) has required seven iterates. An approximate image obtained after three iterates using the algorithm (17) is presented in figure 1(d). Algorithms (13) and (14) have yielded approximations practically coincident with the original image. On figures 2(a) and 2(b) the error value dependences on the iterate number k are shown for the algorithms (13), (14) and (17).

7. Conclusions

We have investigated the restorative approach to the problem of function reconstruction on its line integrals with known weight function. This approach consists of treating the attenuated projections by the Radon transform inversion formula and considering the result of the inversion as a distorted image. It has been found the distortion may be present as an operator series of products and Fourier transforms. This allowed us both to estimate the distortion operator norm and the derive several simple iterative restoration algorithms including a generalized Chang's method. Numerical simulation has shown the advantages of this method in terms of accuracy and efficiency. Its modification (17) may be applied for the fast restoration if either the weight function has a few harmonics in series (5) or computation is performed using a specialized FFT processor.

It seems that estimate (11) could be improved because the above algorithms have converged under numerical simulations for m and M values essentially greater than unity.

References

[1] Herman G T 1980 *Image Reconstruction from Projections* (New York: Academic)
[2] Tretiak O and Metz C 1980 *SIAM J. Appl. Math.* **39** 341–54
[3] Clough A V and Barret H H 1983 *J. Opt. Soc. Am.* **73** 1590–5

[4] Chang L T 1978 *IEEE Trans. Nucl. Sci* **NS-25** 638–43
[5] Moore S C, Brunelle J A and Kirsch C M 1981 *J. Nucl. Med.* **22** 65
[6] Faber T L, Lewis M H, Corbett J R and Stokely E M 1984 *IEEE Trans. Med. Imag.* **MI-3** 101–7
[7] Birkeland J W and Oss J P 1968 *Appl. Opt.* **7** 1635–9
[8] Natterer F 1986 *The Mathematics of Computerized Tomography* (Stuttgart: Teubner)
[9] Tien C L and Lowder J E 1966 *Int. J. Heat Mass Transfer* **9** 698