Section 16.5: Integration in Cylindrical and Spherical Coordinates

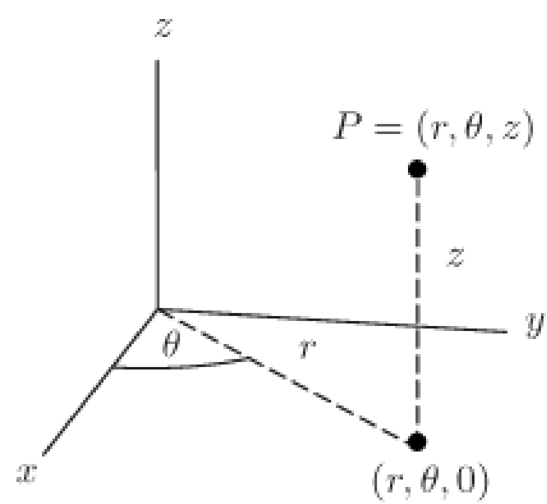
Integration in Cylindrical Coordinates

The cylindrical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are obtained by representing the x and y coordinates using polar coordinates (or potentially the y and z coordinates or x and z coordinates) and letting the third coordinate remain unchanged.

RELATION BETWEEN CARTESIAN AND CYLINDRICAL COORDINATES: Each point in $\mathbb{R}^3$ is represented using $0 \leq r < \infty$, $0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$.

$$x = r\cos\theta,$$
$$y = r\sin\theta,$$
$$z = z.$$

As with polar coordinates in the plane, note that $x^2 + y^2 = r^2$.

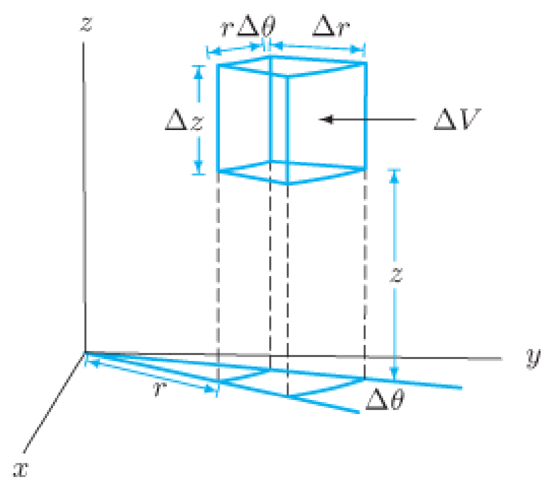


Notice that we can now interpret r as the distance from the point (x, y, z) to the z axis, while the interpretation of θ and z remain unchanged.

Question: What are the surfaces obtained by setting r, θ, and z equal to a constant?

What is dV in Cylindrical Coordinates?

Recall that when integrating in polar coordinates, we set $dA = r\,dr\,d\theta$. When viewing a small piece of volume, ΔV, in cylindrical coordinates, we will see that the correct form for dV is rather intuitive based on this.



It is clear from this image that we should have $\Delta V \approx r\,\Delta r\,\Delta\theta\,\Delta z$. This leads us to the following conclusion:

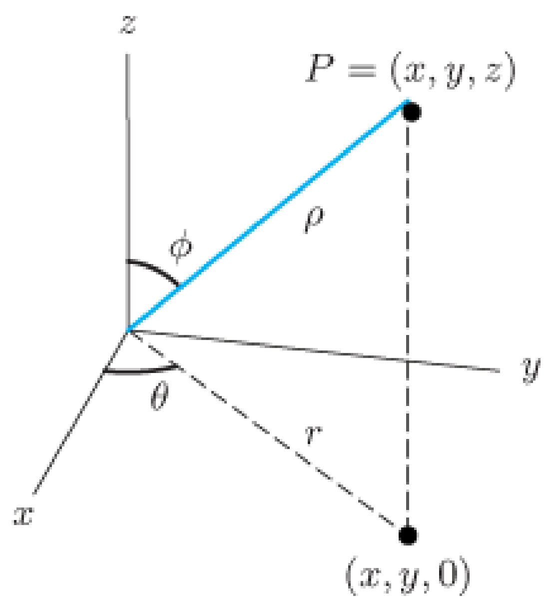
> When computing integrals in cylindrical coordinates, put $dV = r\,dr\,d\theta\,dz$. Other orders of integration are possible.

Examples:

1. Evaluate the triple integral in cylindrical coordinates: $f(x, y, z) = \sin(x^2 + y^2)$, W is the solid cylinder with height 4 with base of radius 1 centered on the z-axis at $z = -1$.

Spherical Coordinates

The spherical coordinates of a point (x, y, z) in $\mathbb{R}^3$ are the analog of polar coordinates in $\mathbb{R}^2$. We define $\rho = \sqrt{x^2 + y^2 + z^2}$ to be the distance from the origin to (x, y, z), θ is defined as it was in polar coordinates, and ϕ is defined as the angle between the positive z-axis and the line connecting the origin to the point (x, y, z).



From the above figure, we can see that $r = \rho \sin \phi$, and $z = \rho \cos \phi$, so using the relationship between Cartesian coordinates (x, y, z) and cylindrical coordinates, $x = r \cos \theta$, $y = r \sin \theta$, $z = z$, we arrive at the following:

RELATIONSHIP BETWEEN CARTESIAN AND SPHERICAL COORDINATES: Each point in $\mathbb{R}^3$ is represented using $0 \leq \rho < \infty$, $0 \leq \phi \leq \pi$, $0 \leq \theta \leq 2\pi$.

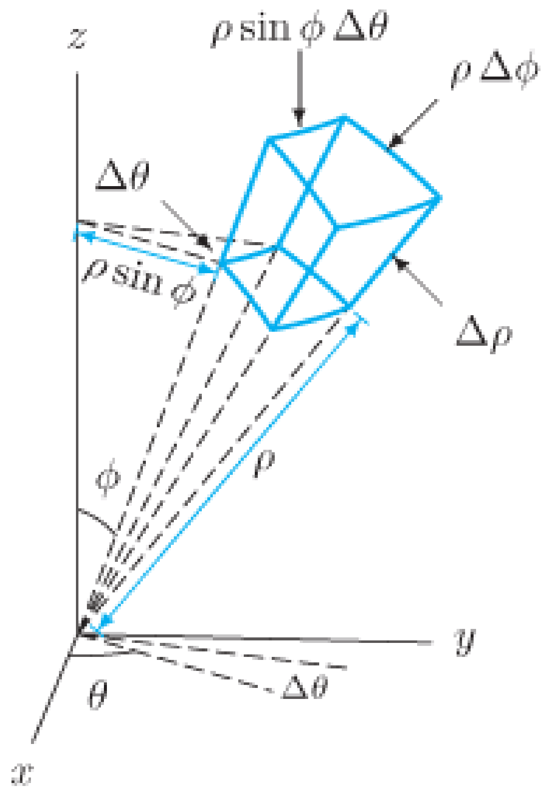
$$x = \rho \sin \phi \cos \theta,$$
$$y = \rho \sin \phi \sin \theta,$$
$$z = \rho \cos \phi.$$

Also, $x^2 + y^2 + z^2 = \rho^2$.

Question: What surfaces are obtained by setting ρ, θ, and ϕ equal to a constant?

What is dV is Spherical Coordinates?

Consider the following diagram:



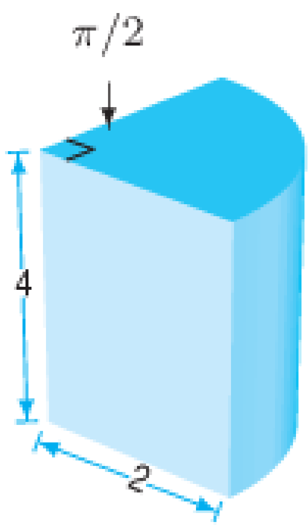
We can see that the small volume ΔV is approximated by $\Delta V \approx \rho^2 \sin\phi\, \Delta\rho\, \Delta\phi\, \Delta\theta$. This brings us to the conclusion about the volume element dV in spherical coordinates:

When computing integrals in spherical coordinates, put $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. Other orders of integration are possible.

Examples:

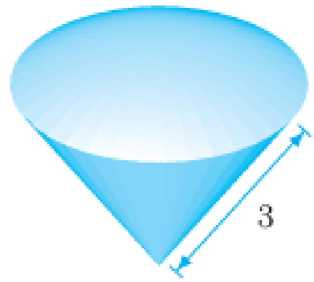
2. Evaluate the triple integral in spherical coordinates. $f(x, y, z) = 1/(x^2 + y^2 + z^2)^{1/2}$ over the bottom half of a sphere of radius 5 centered at the origin.

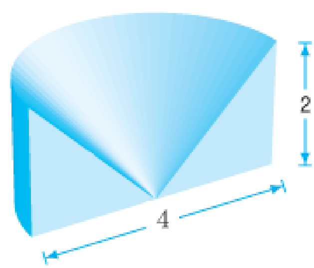
3. For the following, choose coordinates and set up a triple integral, inlcluding limits of integration, for a density function f over the region.



(a)

(b) A piece of a sphere; angle at the center is $\pi/3$.

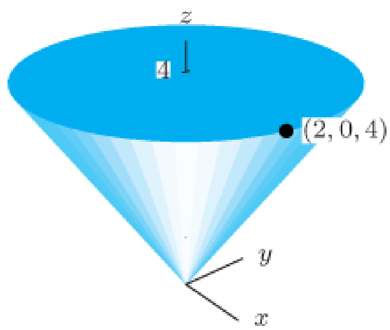




(c)

4. Write a triple integral in spherical coordinates giving the volume of a sphere of radius K centered at the origin. Use the order $d\theta\, d\rho\, d\phi$.

5. If W is the region shown below, what are the limits of integration in the following exercises?



(a) $\displaystyle\int_?^? \int_?^? \int_?^? f(r, \theta, z)\, r\, dz\, dr\, d\theta$

(b) $\displaystyle\int_?^? \int_?^? \int_?^? g(\rho, \phi, \theta)\, \rho^2 \sin\theta\, d\rho\, d\phi\, d\theta$

(c) $\displaystyle\int_?^? \int_?^? \int_?^? h(x, y, z)\, dz\, dy\, dx$

6. Write a triple integral expressing the volume above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere of radius 2 centered at the origin. Do this in both cylindrical and spherical coordinates, including limits of integration.