Written Homework 4

1. Show that if $g(x, y, z)$ is a smooth scalar valued function and $\vec{F}(x, y, z)$ is a smooth vector field, then

$$\operatorname{div}(g\vec{F}) = (\operatorname{grad} g) \cdot \vec{F} + g \operatorname{div} \vec{F}. \tag{1}$$

2. A basic essential property of the electric field $\vec{E}$ is that its divergence is zero at points where there is no charge. Suppose that the only charge is along the z-axis, and that the electric field $\vec{E}$ points radially out from the z-axis and its magnitude depends only on the distance r from the z-axis. Use the Divergence Theorem to show that the magnitude of the field is proportional to $1/r$. [Hint: Consider a solid region consisting of a cylinder of finite length whose axis is the z-axis, and with a smaller concentric cylinder removed.]

3. Show that

$$\operatorname{curl}(\phi\vec{F}) = \phi \operatorname{curl} \vec{F} + (\operatorname{grad} \phi) \times \vec{F} \tag{2}$$

for a smooth scalar function ϕ and a vector field $\vec{F}$.

4. At all points in 3-space $\operatorname{curl} \vec{F}$ points in the direction of $\hat{\imath} - \hat{\jmath} - \hat{k}$. Let C be a circle in the yz-plane, oriented clockwise when viewed from the positive x-axis. Is the circulation of $\vec{F}$ around C positive, zero, or negative?

5. (a) Find $\operatorname{curl}(x^3\hat{\imath} + \sin(y^3)\hat{\jmath} + e^{z^3}\hat{k})$.
   (b) What does your answer to part (a) tell you about $\int_C (x^3\hat{\imath} + \sin(y^3)\hat{\jmath} + e^{z^3}\hat{k}) \cdot d\vec{r}$, where C is the circle $(x - 10)^2 + (y - 20)^2 = 1$, oriented clockwise?
   (c) If C is any closed curve, what can you say about $\int_C (x^3\hat{\imath} + \sin(y^3)\hat{\jmath} + e^{z^3}\hat{k}) \cdot d\vec{r}$?