Section 2.3: Polynomial and Rational Functions

POLYNOMIAL FUNCTION: A *polynomial function* of degree n is defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where a_n, a_{n-1}, …, a_1, a_0 are real numbers, called *coefficients*, with $a_n \neq 0$. The number a_n is called the *leading coefficient*.

Review: Consider the function $f(x) = -3x^4 + 2x^2 + 17$.

(a) Is it a polynomial function?

(b) What is the degree of f?

(c) What is the leading coefficient of f?

(d) What is the basic shape of f? How many turning points do we expect f to have? What is the end behavior of f (what happens to f as x gets large in the positive and negative directions)?

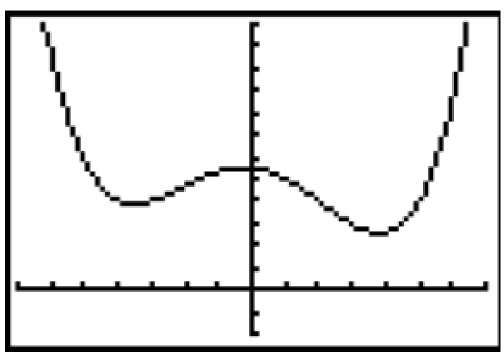
PROPERTIES OF POLYNOMIAL FUNCTIONS:

- A polynomial function of degree n can have at most $n-1$ turning points. Conversely, if the graph of a polynomial function has n turning points, it must have degree at least $n+1$.
- In the graph of a polynomial function of even degree, both ends go up or both ends go down. For a polynomial function of odd degree, one end goes up and one end goes down.
- If the graph goes up as x becomes a large positive number, the leading coefficient must be positive. If the graph goes down as x becomes a large positive number, the leading coefficient is negative.

Preliminaries: Illustrate the four types of end behaviors for polynomials.

Examples:

1. Pictured below is the graph of a polynomial. Give the possible degree of the polynomial, and give the sign (positive or negative) of the leading coefficent.



Rational Functions

RATIONAL FUNCTION: A *rational function* is defined by

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x)$ and $q(x)$ are polynomial functions and $q(x) \neq 0$.

ASYMPTOTES:

- If a function gets larger and larger in magnitude without bound as x approaches the number k, then the line $x = k$ is a *vertical asymptote*.

- If the values of y approach a number k as $|x|$ gets larger and larger, the line $y = k$ is a *horizontal asymptote*.

FINDING ASYMPTOTES AND HOLES: Let $f(x) = \dfrac{p(x)}{q(x)}$ be a rational function. **Vertical Asymptotes and Holes:** Reduce the rational function so that the numerator and the denominator have no common factors. Let $\dfrac{p_0(x)}{q_0(x)}$ denote the reduced rational function.

- The zeros of $q_0(x)$ are vertical asymptotes of $f(x)$.
- Any zeros of $q(x)$ that are not zeros of $q_0(x)$ are holes in the graph of $f(x)$.

Horizontal Asymptotes: Determine the degree of both the numerator $p(x)$ and the denominator $q(x)$.

- If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of $f(x)$.
- If the degree of $p(x)$ is equal to the degree of $q(x)$, then $y = \dfrac{a}{b}$ is the horizontal asymptote of $f(x)$ - where a is the leading coefficient of $p(x)$ and b is the leading coefficient of $q(x)$.
- If the degree of $p(x)$ is more than the degree of $q(x)$, then $f(x)$ does not have a horizontal asymptote.

Examples

2. Graph $f(x) = \dfrac{3x^2 + 6x - 9}{x^2 - x - 12}$. Identify any holes and asymptotes.

3. Suppose a cost-benefit model is given by

$$C(x) = \frac{6.5x}{100 - x},$$

where C is the cost in thousands of dollars of removing x percent of a certain pollutant.

(a) Find the cost of removing 50%, 80%, and 95% of the pollutant.

(b) Graph the function.

(c) According to this model, is it possible to remove all the pollutant?