Section 7.1: Antiderivatives

Up until this point in the course, we have mainly been concerned with one general type of problem: Given a function $f(x)$, find the derivative $f'(x)$. In this section, we will be concerned with solving this problem in the reverse direction. That is, given a function $f(x)$, find a function $F(x)$ such that $F'(x) = f(x)$.

This is called *antidifferentiation*, and any such function $F(x)$ will be called an *antiderivative* of $f(x)$.

DEFINITION: If $F'(x) = f(x)$, then $F(x)$ is called an *antiderivative* of $f(x)$.

Example: If $f(x) = 20x^4$, find a function $F(x)$ such that $F'(x) = f(x)$.

Is the above answer unique?

Write down a formula for *every* antiderivative of $f(x) = 20x^4$.

Examples:

1. Find an antiderivative for the following functions.

   (a) $f(x) = x^4$

(b) $f(x) = x^{15}$

(c) $f(x) = \dfrac{1}{x^2}$

(d) $f(x) = \sqrt{x}$

Indefinite Integral:

INDEFINITE INTEGRAL: If $F'(x) = f(x)$, we can write this using *indefinite integral* notation as

$$\int f(x)\,dx = F(x) + C,$$

where C is a constant called the *constant of integration.*

Note: We call the family of antiderivatives $F(x) + C$ the *general antiderivative* of $f(x)$.

Antiderivatives and Indefinite Integrals:

Let us develop some rules for evaluating indefinite integrals (antiderivatives). The first will be the *power rule* for antiderivatives. This is simply a way of undoing the power rule for derivatives.

POWER RULE: For any real number $n \neq 1$,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C,$$

or

$$\int x^n\,dx = \frac{1}{n+1} \cdot x^{n+1} + C.$$

1. Find the following indefinite integrals.

   (a) $\int x^5\,dx$

   (b) $\int \frac{1}{\sqrt[3]{x}}\,dx$

CONSTANT MULTIPLE AND SUM OR DIFFERENCE RULES: If all indicated integrals exist,

$$\int k \cdot f(x)\,dx = k \int f(x)\,dx,$$

for any real number k, and

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

The constant multiple and sum or difference rules, combined with the power rule for antiderivatives, will allow us to find the antiderivatives of a plethora of different combinations of power functions.

Examples:

2. Find the following indefinite integrals.

(a) $\displaystyle\int \left(17x^{5/2} - \frac{4}{x^3} + 5\right)\,dx$

(b) $\displaystyle\int 4x(x^2 + 2x)\,dx$

Indefinite Integrals of Exponential Functions:

Exercise: Given your knowledge of derivatives, try to find the following indefinite integrals without a formula.

(a) $\int e^x \, dx$

(b) $\int e^{kx} \, dx$

(c) $\int 2^x \, dx$

(d) $\int 3^{kx} \, dx$

This kind of thinking leads us to easily determine the proper formulas for the indefinite integrals of exponential functions.

INDEFINITE INTEGRALS OF EXPONENTIAL FUNCTIONS:

- $\displaystyle\int e^x\,dx = e^x + C$
- $\displaystyle\int e^{kx}\,dx = \frac{e^{kx}}{k} + C$
- $\displaystyle\int a^x\,dx = \frac{a^x}{\ln(a)} + C$
- $\displaystyle\int a^{kx}\,dx = \frac{a^{kx}}{k\ln(a)} + C$

In the above formulas, we are assuming that $k \neq 0$ and that $a > 0$ with $a \neq 1$.

Examples:

3. Find the following indefinite integrals.

   (a) $\displaystyle\int 3e^x\,dx$

   (b) $\displaystyle\int 4e^{-0.2x}\,dx$

(c) $\displaystyle\int (e^{2u} + 4u)\, du$

There is one final antiderivative property we need. What happens if we attempt to use the power rule to fine $\displaystyle\int x^{-1}\, dx$?

INDEFINITE INTEGAL OF $\dfrac{1}{x}$:

$$\int x^{-1}\, dx = \int \frac{1}{x}\, dx = \ln|x| + C.$$

Examples:

4. Find the following indefinite integrals.

   (a) $\displaystyle\int \frac{3}{x}\, dx$

   (b) $\displaystyle\int \left(\frac{9}{x} - 3e^{-0.4x}\right) dx$

   (c) $\displaystyle\int \frac{\sqrt{x}+1}{\sqrt[3]{x}}\, dx$

Specific Antiderivatives (Solving for C):

Examples:

5. If $f(x) = e^{2x} - 9x^2$ and $F(0) = 4$, find $F(x)$, the specific antiderivative of $f(x)$.

6. Find the cost function if the marginal cost function is
$$C'(x) = x + \frac{1}{x^2}$$
and 2 units cost \$5.50.

7. Find the demand function if the marginal revenue is

$$R'(x) = 500 - 0.15\sqrt{x}.$$