Section 7.3: Area and the Definite Integral

So far in chapter 7 we have learned various techniques for computing *indefinite integrals* of the form $\int f(x)\,dx$. The notation $\int \ldots\,dx$ always meant "find the antiderivative of what is in between".

Moving forward, we will learn a new type of integral, called the *definite integral.* The notation for such an integral is $\int_a^b f(x)\,dx$, and these definite integrals are fundamentally different from indefinite integrals. Computing the definite integral means computing a single number, whereas finding indefinite integrals meant finding functions.

An Introductory Example:

The distance from Tucson to Mesa, AZ is approximately 120 miles.

1. If I make this trip at a *constant velocity*, and it takes me 2 hours, how fast am I traveling? Sketch a graph of velocity vs. time (velocity on the vertical axis and time on the horizontal axis).

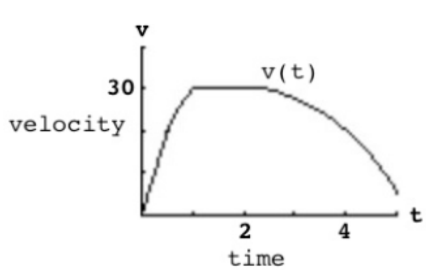
Question: What is the area of the rectangle formed by the graph?

2. Now suppose that instead of driving the same speed for the entire trip, I first drive 40 mph for 3/4 of an hour, then I get on a highway and drive 70 mph for 1 hour, and then I get off of the highway and drive 40 mph again for 1/2 an hour. Sketch the velocity vs. time graph.

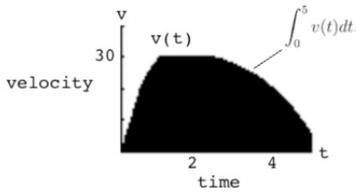
Question: How far did I travel? Did I make it to Mesa?

The fact that the area under the velocity curve gives us the total distance that we traveled is not an accident. It happens to be a feature, and it is a feature that it turns out we can exploit with calculus. It also turns out to be true for any positive velocity function.

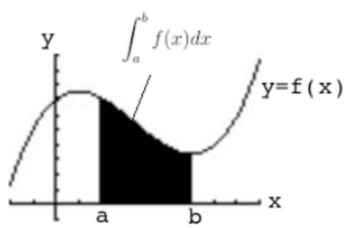
Consider the velocity function whose graph is shown below.



The area under this curve gives the total distance traveled from time $t = 0$ to $t = 5$. The area under the velocity curve is called the *definite integral* of $v(t)$ from $t = 0$ to $t = 5$, and is denoted by $\int_0^5 v(t)\,dt$:



The techniques that we will learn can be used to compute the area underneath any curve given by a function $y = f(x)$. If $y = f(x)$, then the (signed) area under the function from $x = a$ to $x = b$ is given by $\int_a^b f(x)\,dx$

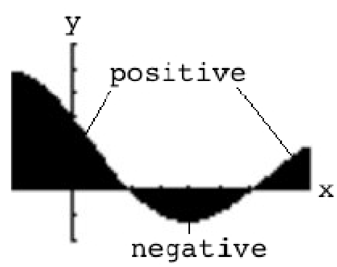


In this section, we will not learn the shortcuts for computing definite integrals. Instead, we will learn how they are built and how we can approximate them using rectangles. We will also be able to compute definite integrals in the case where our graphs form common shapes like triangles and circles.

First, let us pose the following question.

Question: What happens when the graph of a velocity function becomes negative? How should you interpret the definite integral in that case?

In general, if $y = f(x)$, the part above the x-axis is counted positively, and the part below the x-axis is counted negatively. And thus, the definite integral gives a signed area.



Definition of the Definite Integral:

The textbook will define the definite integral as follows:

THE DEFINITE INTEGRAL: If $f(x)$ is defined on the interval $[a, b]$, the definite integral of f from $x = a$ to $x = b$ is given by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x,$$

provided the limit exists, where $\Delta x = (b - a)/n$, and x_i is any x-value in the i^{th} interval.

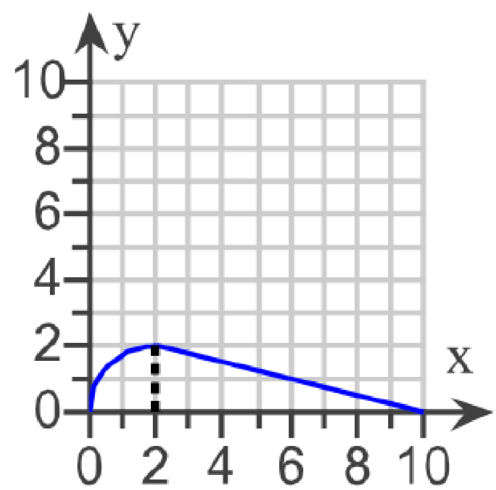
Let us just say one thing about the summation notation. The Σ used in the notation is a convenient way to indicate that we are adding a bunch of numbers together. In this case,

$$\sum_{i=1}^{n} f(x_i)\,\Delta x = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \cdots + f(x_n)\Delta x.$$

Each term of the form $f(x_i)\Delta x$ in the sum represents the area of a rectangle with width $f(x_i)$ and length Δx.

Examples:

3. Find $\int_0^{10} f(x)\,dx$ for the graph of $y = f(x)$, where $f(x)$ consists of line segments and circular arcs.



4. Find the exact value of the integral using formulas from geometry.

$$\int_0^4 \sqrt{16 - x^2}\,dx$$

5. The graph of a function f is given below. Approximate the area under the curve from $x = 0$ to $x = 3$ with rectangles, using the following methods with $n = 3$ (three rectangles).

   (a) Use left endpoints (LHS)

   (b) Use right endpoints (RHS)

   (c) Average the LHS and RHS.

   (d) Use midpoints.