Section 7.4: The Fundamental Theorem of Calculus

In this section we will see how the two types of integrals we have learned are connected. The indefinite integral is just a way of talking about antiderivatives. The definite integral means calculating an area under a curve. The Fundamental Theorem shows us that finding antiderivatives is essential to being able to calculate the area under the curve.

THE FUNDAMENTAL THEOREM OF CALCULUS (FTC): Let $f(x)$ be a continuous function on the interval $[a, b]$, and suppose F is any antiderivative of f. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Examples:

1. Evaluate the definite integral $\int_{-3}^{3} (9 - x^2)\,dx$

2. Evaluate the following definite integrals using the FTC.

(a) $\displaystyle\int_4^9 \sqrt{x}\,dx$

(b) $\displaystyle\int_1^2 \frac{9}{x^4}\,dx$

(c) $\displaystyle\int_0^1 (5e^x + 6x)\,dx$

(d) $\displaystyle\int_1^3 \frac{1}{t}\,dt$

We will now see that we can apply to FTC to definite integrals when finding the antiderivative requires the substitution method. There are two ways to go about this, and we will do both with the following example.

Examples:

3. Evaluate $\int_0^1 x(x^2+1)^3\,dx$ using the following two approaches.

   (a) First fine the antiderivative of $x(x^2+1)^3$, and then apply the fundamental theorem using $x=0$ and $x=1$ as the limits of integration.

   (b) After determining your substitution, change the limits of integration to reflect your new variable.

4. Evaluate the following definite integrals.

(a) $\displaystyle\int_0^2 x^2\sqrt{1+x^3}\,dx$

(b) $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$

(c) $\displaystyle\int_0^3 \frac{6x}{x^2+1}\,dx$

(d) $\displaystyle\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

5. A small company of science writers found that its rate of profit (in thousands of dollars per year) after t years of operation is given by the function
$$P'(t) = (6t + 12)(t^2 + 4t + 4)^{1/4}.$$
Find the total profit accumulated over the first four years.