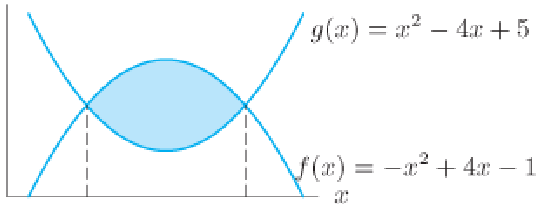
Section 7.5: The Area Between Two Curves

Suppose we wanted to calculate the area of the shaded region shown below:



One can see that the are of the region is going to be the area under $f(x)$ minus the area under $g(x)$ over the relevant interval. We can formalize this result as follows:

THE AREA BETWEEN TWO CURVES: If f and g are continuous and $f(x) \geq g(x)$ on the interval $[a, b]$, then the area between the curves $f(x)$ and $g(x)$ from $x = a$ to $x = b$ is given by

$$\int_a^b (f(x) - g(x))\, dx.$$

Note: There are times where the functions $f(x)$ and $g(x)$ intersect over the interval $[a, b]$ and the roles of top and bottom become reversed. In these kinds of situation, you can often get away with a special trick (especially if you are able to use a calculator to solve the problem): The area between the two curves can always be calculated according to

$$\int_a^b |f(x) - g(x)|\, dx.$$

Examples:

1. Find the area between the curves $y = 4 - x^2$ and $y = x + 2$ on the interval $[-2, 1]$.

2. A construction company has an expenditure rate of $E'(x) = e^{0.14x}$ dollars per day on a particular paving job and an income rate of $I'(x) = 133.9 - e^{0.14x}$ dollars per day on the same job, where x is the number of days after the start of the job. The company's profit on the job will equal total income minus total expenditures. Profit will be maximized if the job ends at the optimum time, which is the point where the two rate curves meet.

   (a) Find the optimum number of days for the job to last

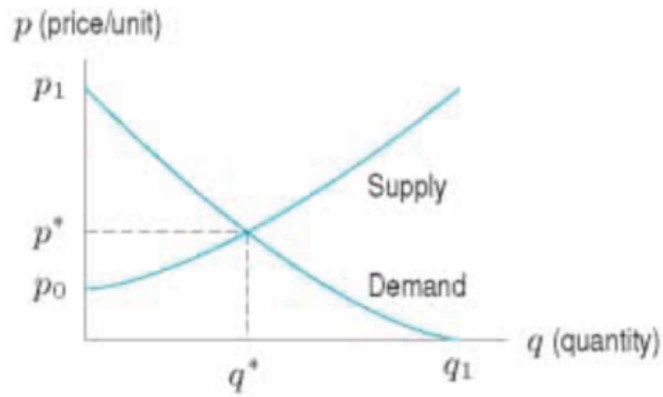
   (b) Find the total income for the optimum number of days.

   (c) Find the total expenditures for the optimum number of days.

   (d) Find the maximum profit for the job.

Consumer and Producer Surplus

Recall what a typical supply and demand system looks like:



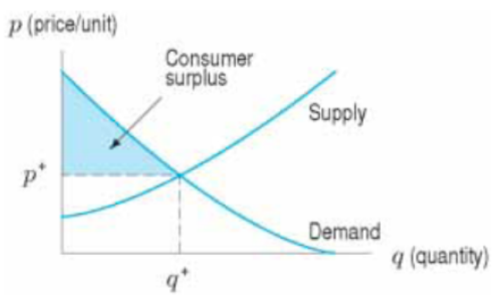
(a) What do the supply and demand curves mean?

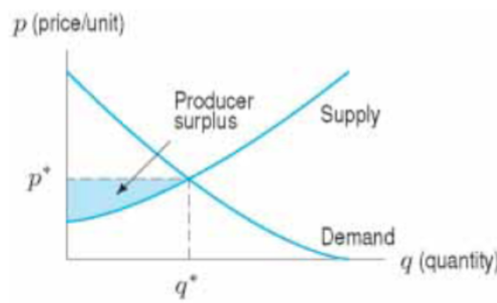
(b) What is the significance of the point (q^*, p^*)?

CONSUMER AND PRODUCER SURPLUS: Suppose $p = D(q)$ is a demand curve and $p = S(q)$ is a supply curve. If (q_0, p_0) represents the equilibrium point, we have the following.

1. The*consumer surplus* is defined to be the area between $p = D(q)$ and $p = p_0$ from $q = 0$ to $q = q_0$:
$$C.S. = \int_0^{q_0} (D(q) - p_0)\, dq.$$

2. The *producer surplus* is defined to be the area between $p = p_0$ and $p = S(q)$ from $q = 0$ to $q = q_0$:
$$P.S. = \int_0^{q_0} (p_0 - S(q))\, dq.$$





Examples:

3. The demand curve for a product has equation $p = 20e^{-0.002q}$ and the supply curve has equation $p = 0.02q + 1$ for $0 \leq q \leq 1000$, where q is quantity and p is price in \$/unit.

   (a) Which is higher, the price at which 300 units are supplied or the price at which 300 units are demanded? Find both prices.

   (b) Sketch the supply and demand curves. Find the equilibrium price and quantity.

   (c) Calculate the consumer and producer surplus.

4. The demand curve for a product has equation $p = 35 - q^2$ and the supply curve has equation $p = 3 + q^2$ for $0 \leq q \leq 6$, where q is quantity and p is price in \$/unit.

   (a) Sketch the supply and demand curves. Find the equilibrium price and quantity.

   (b) Calculate the consumer and producer surplus.