Section 2.2: The Derivative at a Point

It turns out that the concept of instantaneous rate of change is not exclusive to velocity functions. In fact, we can define both the average rate of change and the instantaneous rate of change for any arbitrary function $f(x)$.

> The *average rate of change* for a function $f(x)$ over the interval from $x = a$ to $x = a + h$ is given by the expression
> $$\frac{f(a+h) - f(a)}{h}$$

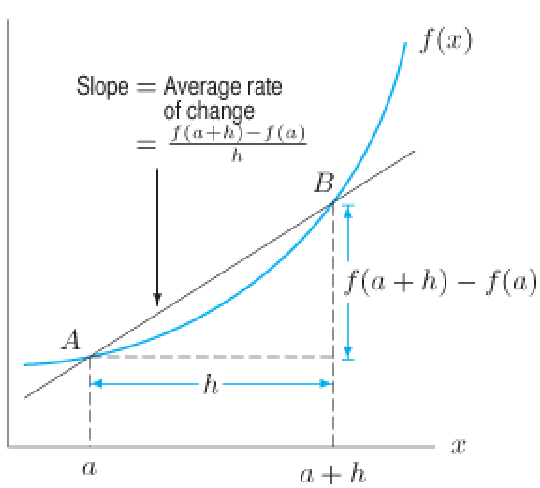
Even though the independent variable no longer necessarily represents time, we will still refer to the above expression as a rate of change. If we want to emphasize the independent variable, x, we will refer to it as the average rate of change of f *with respect to* x.

Instantaneous Rate of Change: The Derivative

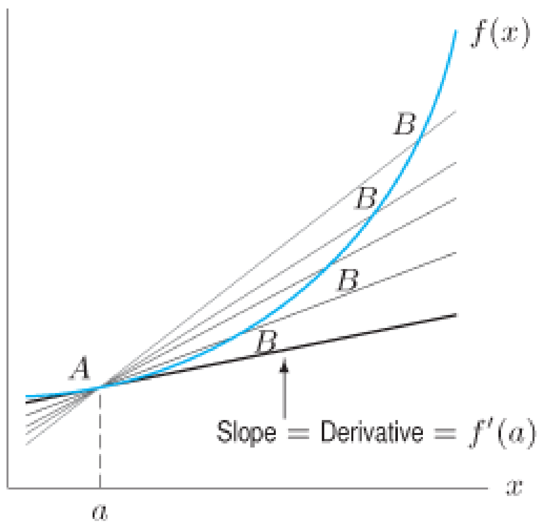
> The *derivative* of $f(x)$ at $x = a$, written $f'(a)$, is defined as the instantaneous rate of change of $f(x)$ at $x = a$:
> $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
> If the limit exists, then f is said to be *differentiable* at $x = a$.

Visualizing the Derivative: Slope of Curve and Slope of Tangent

As with average velocity, in general the average rate of change of a function can be visualized as the slope of the secant line connecting $(a, f(a))$ and $(a + h, f(a + h))$:



As h approaches zero, the secant line approaches the *tangent line* to $f(x)$ at $x = a$:



Examples:

1. The table shows values of $f(x) = x^3$ near $x = 2$ (to three decimal places). Use it to estimate $f'(2)$.

x	1.998	1.999	2.000	2.001	2.002
$f(x)$	7.976	7.988	8.000	8.012	8.024

2. By choosing small values of h, estimate the instantaneous rate of change of $f(x) = x^3$ with respect to x at $x = 1$.

3. Label points A, B, C, D, E, and F on the graph of $y = f(x)$ in the figure below.

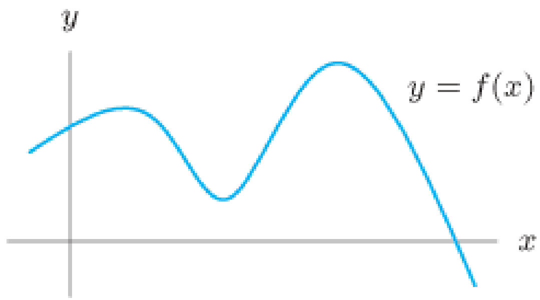
   (a) Point A is a point on the curve where the derivative is negative.

   (b) Point B is a point on the curve where the value of the function is negative.

   (c) Point C is a point on the curve where the derivative is largest.

   (d) Point D is a point on the curve where the derivative is zero.

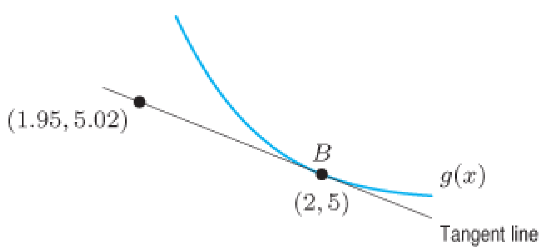
   (e) Points E and F are different points on the curve where the derivative is about the same.



4. Use the figure below to find the following quantities:

   (a) $g(2)$

   (b) $g'(2)$.



5. Let $f(x) = x^2$. Find $f'(2)$ using the limit definition of the derivative.

6. Let $g(x) = \dfrac{1}{x}$. Find $g'(3)$ using the limit definition of the derivative.

7. Find an equation for the tangent line of $f(x) = x^2 + x$ at $x = 3$. Sketch the function and this tangent line.