Section 2.4: Interpretations of the Derivative

Given a function $f(x)$, the derivative, $f'(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If we think of h as the change in x-values, Δx, then this can be written as

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

We will define a new notation for the derivative based on this interpretation. This notation was first developed by Gottfried Wilhelm Leibniz in the 17th century, and is therefore called the *Leibniz notation* for the derivative:

LEIBNIZ NOTATION FOR THE DERIVATIVE: If $y = f(x)$, then we can write the derivative $f'(x)$ as

$$f'(x) = \frac{dy}{dx}.$$

One way to remember why this notation is intellectually consistent with our understanding of the derivative is by thinking of it as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

In other words, we imagine that in the limit as $\Delta x \to 0$, the change in y-values also necessarily converge to zero, and the dx can be thought of as an *infinitesimal* change in x (similarly, dy can be thought of as in infinitesimal change in y).

Question: If $y = f(x)$, what are the units of $\frac{dy}{dx}$?

How does one signify that they are evaluating the derivative at a point using Leibniz notation?

This is a disadvantage that the Leibniz notation has. One can signify that they are evaluating the derivative at a specific point by using a vertical bar:

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}.$$

THE DERIVATIVE OPERATOR: A useful aspect of the Leibniz notation is that we can introduce the derivative operator, $\dfrac{d}{dx}$:

$$\frac{d}{dx}\left[f(x)\right] = f'(x).$$

We read the derivative operator d/dx as "the derivative of [blank] with respect to x". To really see how it fits with the Leibniz notation, notice that if we have $y = f(x)$, we can imagine the algebra working as follows:

$$f'(x) = \frac{d}{dx}\left[f(x)\right] = \frac{d}{dx}(y) = \frac{dy}{dx}.$$

But the real purpose of this section is to learn how to interpret the meaning of derivative statements. Let's accomplish this through some examples:

1. An economist is interested in how the price of a certain commodity affects its sales. Suppose that at a price of $\$p$, a quantity q of the commodity is sold. If $q = f(p)$, explain in economic terms the meaning of the statements $f(10) = 240,000$ and $f'(10) = -29,000$.

2. The population of Mexico in millions is $P = f(t)$, where t is the number of years since 1980. Explain the meaning of the statements:

   (a) $f'(6) = 2$

   (b) $f^{-1}(95.5) = 16$

   (c) $(f^{-1})'(95.5) = 0.46$

3. The temperature, H, in degrees Celsius, of a cup of coffee placed on the kitchen counter is given by $H = f(t)$, where t is in minutes since the coffe was put on the counter.

   (a) Is $f'(t)$ positive or negative? Give a reason for your answer.

   (b) What are the units of $f'(20)$? What is its practical meaning in terms of the temperature of the coffee?

4. After investing \$1000 at an annual interest rate of 7% compounded continuously for t years, your balance is \$$B$, where $B = f(t)$. What are the units of dB/dt? What is the financial interpretation of dB/dt?

5. A company's revenue from car sales, C (in thousands of dollars), is a function of advertising expenditure, a, in thousands of dollars, so $C = f(a)$.

   (a) What does the company hope is true about the sign of f'?

   (b) What does the statement $f'(100) = 2$ mean in practical terms? How about $f'(100) = 0.5$?

   (c) Suppose the company plans to spend about \$100,000 on advertising. If $f'(100) = 2$, should the company spend more or less than \$100,000 on advertising? What if $f'(100) = 0.5$?