Section 2.5: The Second Derivative

Since the derivative of a function f is itself a function, we can take its derivative. This is called the *second derivative*. We have

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}.$$

Question: Recall that if $f' > 0$ on an interval, then f is increasing on that interval. Similarly, if $f'' > 0$ on an interval, then f' is increasing on that interval. How does this effect the shape of f?

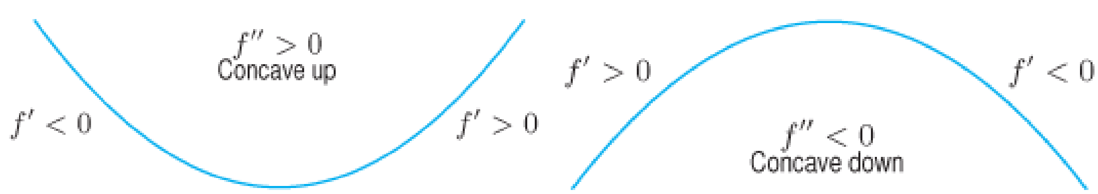
What do the Derivatives Tell Us?

- If $f' > 0$ on an interval, then f is increasing on that interval.
- If $f' < 0$ on an interval, then f is decreasing on that interval.
- If $f'' > 0$ on an interval, then f' is increasing on that interval.
- If $f'' < 0$ on an interval, then f' is decreasing on that interval.

If we think about what it means, graphically, for f' to be increasing or decreasing on an interval, we come to form the following conclusion:

- If $f'' > 0$ on an interval, then f is *concave up* on that interval.
- If $f'' < 0$ on an interval, then f is *concave down* on that interval.

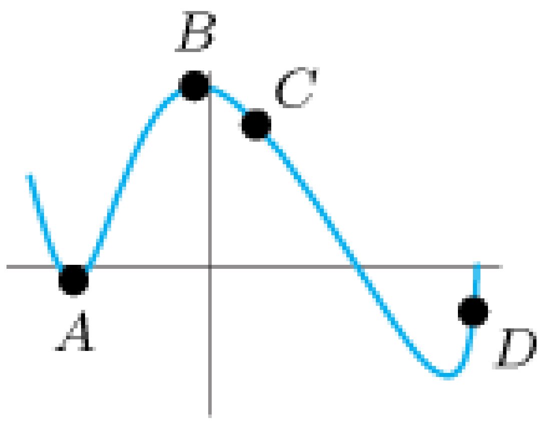
The following graphic illustrates this situation:



Examples:

1. At exactly two of the labeled points in the figure below, the derivative f' is 0; the second derivative f'' is not 0 at any of the labeled points. On a copy of this table, give the signs of f, f', and f'' at each marked point.

Point	f	f'	f''
A			
B			
C			
D			



2. Sketch the graph of a function whose first derivative is everywhere negative and whose second derivative is positive for some x-values and negative for other x-values.

Velocity and Acceleration

Recall that if $s(t)$ represents the position of an object at time t, then $s'(t) = ds/dt$ is the velocity of this object at time t. If the velocity is increasing, we say that the object is *accelerating*. If $v(t)$ is the velocity of the object at time t, then the average acceleration of the object from t to $t+h$ is given by

$$\text{Average Acceleration} = \frac{v(t+h) - v(t)}{h}.$$

INSTANTANEOUS ACCELERATION: If $v(t)$ represents the velocity of an object at time t, the *instantaneous acceleration* of the object is given by $v'(t)$. That is

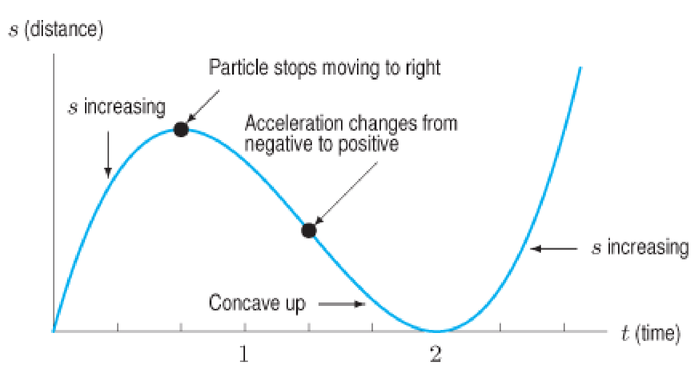
$$\text{Instantaneous Acceleration} = v'(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h}.$$

If we ever use the terms velocity or acceleration on their own, we will assume that we are referring to the instantaneous velocity or acceleration. Since velocity is the derivative of position, we see that acceleration is the second derivative of position.

If $y = s(t)$ is the position of an object at time t, then

- $v(t) = \dfrac{dy}{dt} = s'(t)$.
- $a(t) = \dfrac{d^2y}{dt^2} = s''(t) = v'(t)$.

The following figure illustrates the position of a particle moving in a straight line. The acceleration of the particle is zero only once (for a single instant).



INFLECTION POINTS: A point where the second derivative of a function goes from positive to negative is called an *inflection point*. In other words, an inflection point is a point at which the concavity of the function changes from positive to negative, or from negative to positive.

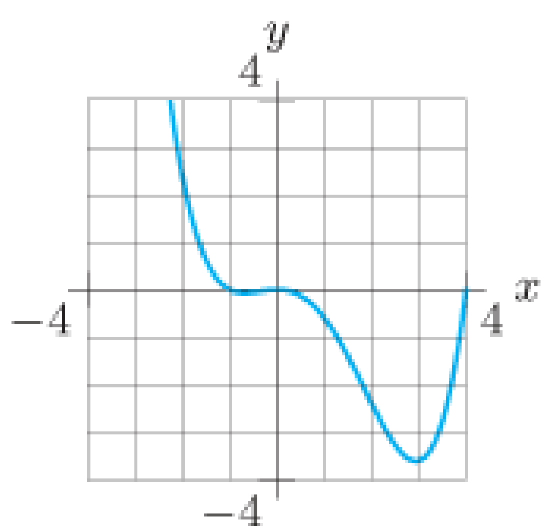
3. An accelerating sports car goes from 0 mph to 60 mph in five seconds. Its velocity is given in the following table, converted from miles per hour to feet per second, so that all measurements are in seconds (Note that 1 mph is 22/15 ft/s).

Time, (t) (sec)	0	1	2	3	4	5
Velocity, $v(t)$ (ft/sec)	0	30	52	68	80	88

(a) Find the average velocity over the first two seconds.

(b) Estimate the acceleration at time $t = 2$ seconds.

4. Sketch the second derivative of the following function.



5. "Winning the war on poverty" has been described cynically as slowing the rate at which people are slipping below the poverty line. Assuming that this is happening

   (a) Sketch a graph of the total number of people in poverty against time.

   (b) If N is the total number of people below the poverty line at time t, what are the signs of dN/dt and d^2N/dt^2? Explain.

6. At which of the marked x-values in the figure below can the following statements be true?

   (a) $f(x) < 0$

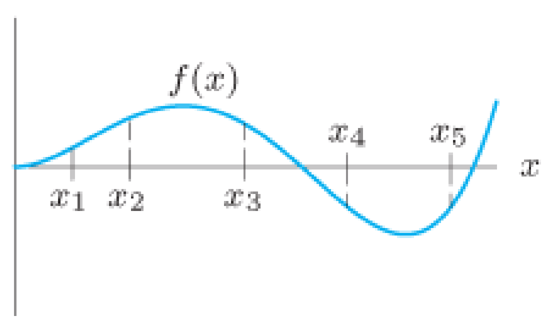
   (b) $f'(x) < 0$

   (c) $f(x)$ is decreasing

   (d) $f'(x)$ is decreasing

   (e) Slope of $f(x)$ is positive

   (f) Slope of $f(x)$ is increasing



7. A function f has $f(5) = 20$, $f'(5) = 2$, and $f''(x) < 0$ for $x \geq 5$. Which of the following are possible values for $f(7)$ and which are impossible?

   (a) 26

   (b) 24

   (c) 22