Section 2.6: Differentiability

What Does it Mean for a Function to be Differentiable?

At a glance, differentiability is simple enough in concept. A function is said to be differentiable at a point if it has a derivative at that point. However, it is vital that we understand that this is a statement about the *existence of a limit*:

DIFFERENTIABILITY: The function f is *differentiable* at x if

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ exists.}$$

Thus, the graph of f has a nonvertical tangent line at x. The value of the limit and the slope of the tangent line are the derivative of f at x.

What are some of the ways that a function can fail to be differentiable at a point?

- The function is not continuous at the point.
- The graph has a sharp corner at that point.
- The graph has a vertical tangent line.

Preliminary Exercise: Sketch a graph of a function that includes points of all of the above types where the function fails to be differentiable.

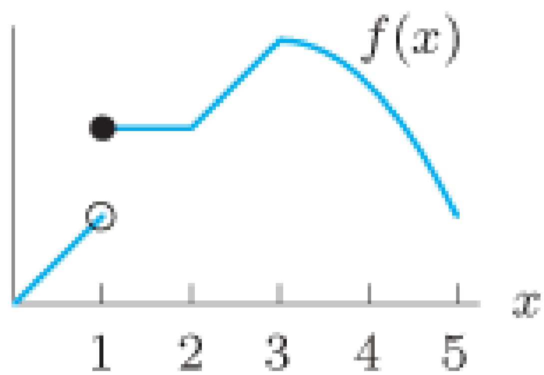
Examples of Non Differentiable Functions: Provide two examples of functions that fail to be differntiable at a point.

Examples:

1. For the graph given below, list the x-values for which the function appears to be

   (a) Not continuous.

   (b) Not differentiable.

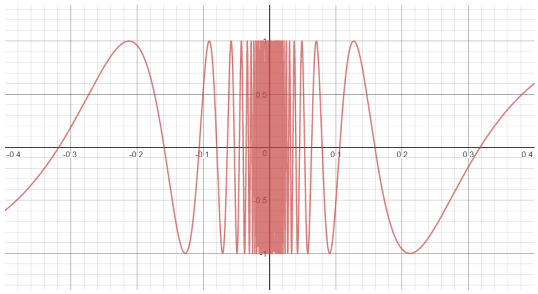


Pathological Example: $\sin\left(\frac{1}{x}\right)$

The function defined by

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0 \end{cases}$$

is an informative example when discussing continuity and differentiability. The graph of this function is pictured below:

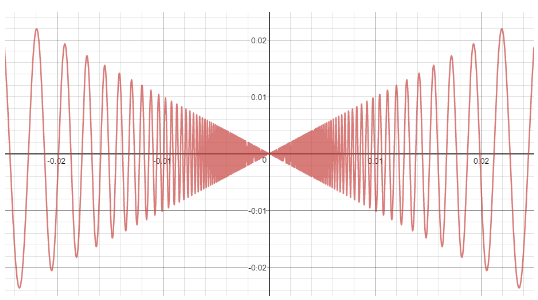


The function above has what is called an *essential discontinuity* at $x = 0$. It actually oscillates an infinite number of times over any interval around $x = 0$ (no matter how small the interval is!). This function is clearly not differentiable at $x = 0$ (why?)

However, if we multiply $\sin\left(\frac{1}{x}\right)$ by powers of x, we get some interesting behavior. Consider the graph of the function

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & x \neq 0, \\ 0 & x = 0 \end{cases}$$

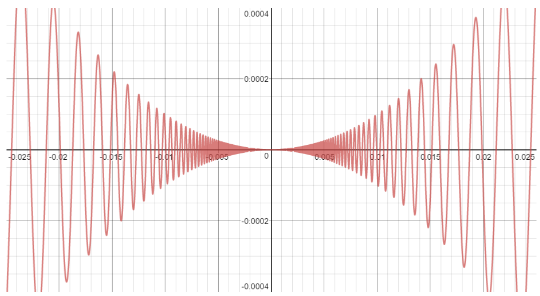
shown below.



Discuss: Discuss the behavior of the above function at $x = 0$. If we define a function $g(x)$ to be equal to zero for $x = 0$ and to be equal to the above function for all other values of x, is $g(x)$ continuous at $x = 0$? Is it differentiable at $x = 0$?

Now, observe what happens when we multiply $\sin\left(\frac{1}{x}\right)$ by x^2 instead of x. Pictured below is the graph of

$$y = x^2 \sin\left(\frac{1}{x}\right)$$



Question: Let us define

$$g(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Is $g(x)$ differentiable at $x = 0$? Use the limit definition of the derivative to check that the limit exists. It is okay to assume that

$$\lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0.$$

Examples:

2. The acceleration due to gravity, g, varies with height above the surface of the earth in a certain way. If you go down below the surface of the earth, g varies in a different way. It can be shown that, according to Newton's graviational formulation, g is given by

$$g = \begin{cases} \frac{GMr}{R^3} & \text{for } r < R \\ \frac{GM}{r^2} & \text{for } r \geq R \end{cases}$$

   (a) Sketch a graph of g against r.

   (b) Is g a continuous function of r? Explain.

   (c) Is g a differentiable function of r? Explain

4. Graph the function defined by

$$g(r) = \begin{cases} 1 + \cos(\pi r/2) & \text{for } -2 \le r \le 2 \\ 0 & \text{for } r < -2 \text{ or } r > 2 \end{cases}$$

(a) Is g continuous at $r = 2$? Explain your answer.

(b) Do you think that g is differentiable at $r = 2$? Explain your answer.