Section 3.4: The Chain Rule

The purpose of this section is to understand how the rate of change of a composition of functions of the form $y = f(g(x))$ behaves. To do this, consider the example given in the textbook:

Hot Air Balloon Example: Imagine you are moving straight upwards in a hot air balloon. Your height, y, is given as a function of time, $y = g(t)$. As you move upwards, the air temperature surrounding you is changing, so the air temperature can be modeled as a function of height, $H = f(y)$.

If the air temperature decreases by 16° F for every mile you climb into the sky, and if you are traveling upwards at 2 mph, what is the rate at which air pressure is changing with respect to time?

From this, we arrive at a solid conclusion based on a very intuitive understanding of how rates of change work in the physical world. The rate of change of a composite function is the product of the rate of change of the outside function and the rate of change of the inside function

To see how this makes sense mathematically, let $z = g(x)$ and $y = f(z)$, so that $y = f(g(x))$. If we look at the average rates of change, we have

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta z} \cdot \frac{\Delta z}{\Delta x}.$$

Since $\dfrac{dy}{dx} = \displaystyle\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$, we have

THE CHAIN RULE:

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}.$$

Using function notation, we can write the chain rule as follows:

THE CHAIN RULE: If f and g are differentiable functions, then

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Examples:

1. Find the derivative of the following functions.

   (a) $f(x) = e^{x^2}$

   (b) $w = (t^3 + 1)^{100}$

   (c) $f(x) = \sqrt{1 - x^2}$

   (d) $k(x) = (x^3 + e^x)^4$

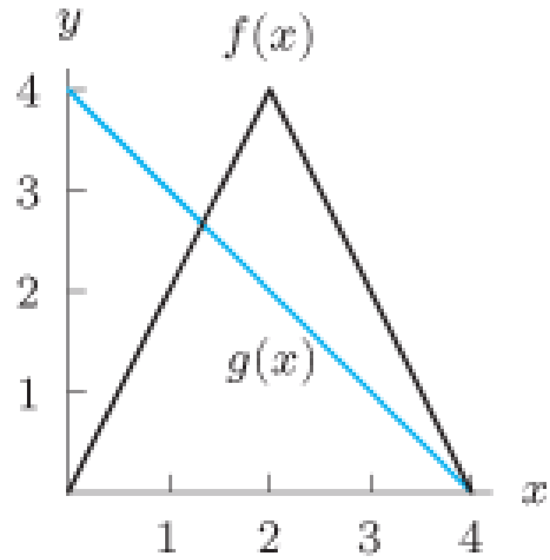
(e) $g(x) = e^{\pi x}$

(f) $f(w) = (5w^2 + 3)e^{w^2}$

(g) $h(z) = \left(\dfrac{b}{a + z^2} \right)^4$, where a and b are constants.

(h) $f(t) = 2e^{-2e^{2t}}$

2. Let $h(x) = f(g(x))$. Using the figure below, find the values of the following:



(a) $h'(1)$

(b) $h'(2)$

(c) $h'(3)$

3. Let $f(x) = 6e^{5x} + e^{-x^2}$. Find an equation for the tangent line to $y = f(x)$ at $x = 1$.

4. For what values of x is the graph of $y = e^{-x^2}$ concave down?

5. If the derivative of $y = k(x)$ equals 2 when $x = 1$, what is the derivative of

   (a) $k(2x)$ when $x = 1/2$?

   (b) $k(x+1)$ when $x = 0$?

   (c) $k\left(\frac{1}{4}x\right)$ when $x = 4$?