Section 3.6: The Chain Rule and Inverse Functions

The Derivative of $x^{1/2}$

We have already seen how to take the derivative of $x^{1/2}$ using the power rule. However, now that we have the chain rule at our disposal, we can take an alternative approach. Notice that if $f(x) = x^{1/2}$, we have

$$(f(x))^2 = x$$

Next, we will use the chain rule to take the derivative of both sides of the equation and solve for $f'(x)$.

$$\Rightarrow \frac{d}{dx}(f(x))^2 = \frac{d}{dx}(x)$$

$$\Rightarrow 2(f(x)) \cdot f'(x) = 1$$

$$\Rightarrow f'(x) = \frac{1}{2f(x)}$$

$$\Rightarrow f'(x) = \frac{1}{2x^{1/2}}$$

Thus we have shown that we get the same derivative for $x^{1/2}$ as we would have using the power rule.

Problem: Use the relationship $e^{\ln x} = x$ to derive a formula for $\frac{d}{dx}(\ln x)$.

DERIVATIVE OF $\ln x$. If $x > 0$, we have

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

After completing your worksheet, we have obtained the following formulas for derivatives of specific types of inverse functions.

DERIVATIVES OF INVERSE FUNCTIONS:

- $\dfrac{d}{dx}(\arctan x) = \dfrac{1}{1+x^2}$
- $\dfrac{d}{dx}(\arcsin x) = \dfrac{1}{\sqrt{1-x^2}}$
- $\dfrac{d}{dx}(f^{-1}(x)) = \dfrac{1}{f'(f^{-1}(x))}$

Question: Using the same technique, find the derivative of $\log_a x$, where a is any positive number such that $a \neq 1$.

Examples:

1. Find the derivative.

   (a) $f(x) = \ln(1 - x)$

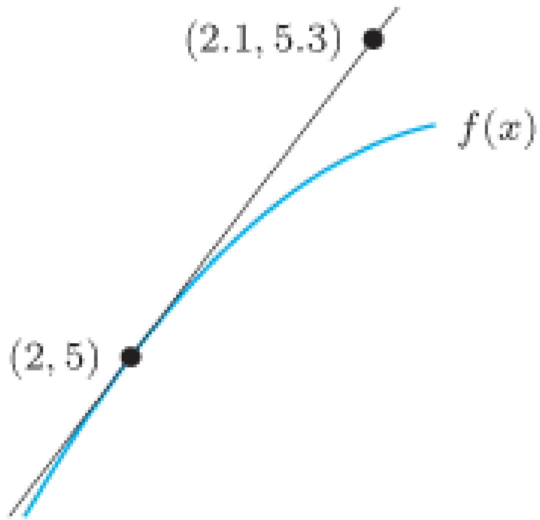
   (b) $f(x) = \arctan(3x)$

   (c) $f(a) = \ln(\sin a)$

   (d) $f(y) = \arcsin(y^2)$

(e) $T(u) = \arctan\left(\dfrac{u}{1+u}\right)$

2. Using the figure below, calculate the derivative.



(a) $h'(2)$ if $h(x) = (f(x))^3$

(b) $k'(2)$ if $k(x) = (f(x))^{-1}$

(c) $g'(5)$ if $g(x) = f^{-1}(x)$