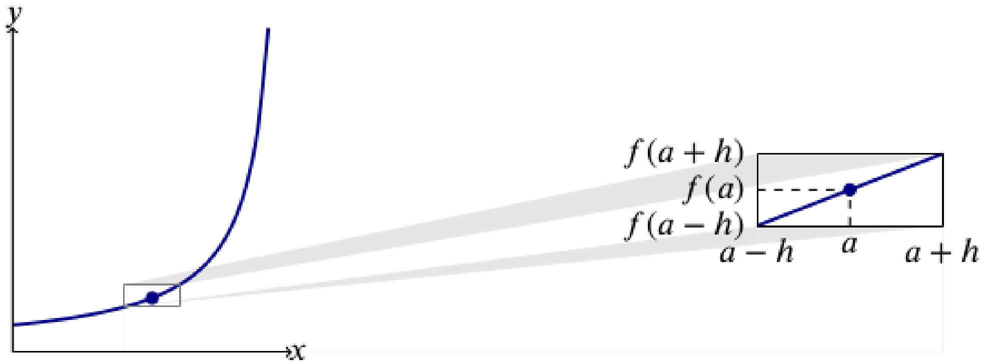
Section 3.9: Linear Approximation and the Derivative

The Tangent Line Approximation

One of the fundamental tenants of calculus is the concept of *local linearity*. The concept is actually rather simple. If f is a differentiable function, then if we zoom into the graph of $y = f(x)$ at a point $x = a$, the graph begins to closely resemble the tangent line to $f(x)$ at $x = a$. So much so, in fact, that it would be impossible to distinguish a difference with the naked eye.



This suggests that for values of x very close to a, one can get very good approximations for the value of $f(x)$ by using the tangent line at $x = a$:

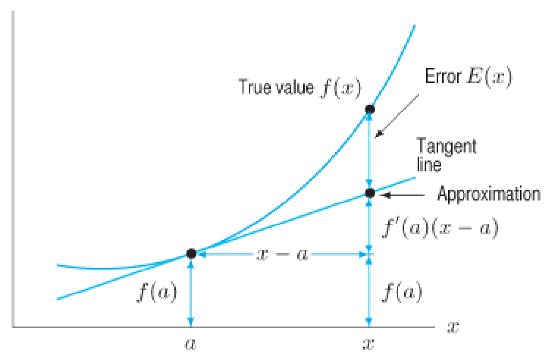
THE TANGENT LINE APPROXIMATION: Suppose f is differentiable at $x = a$. Then, for values of x near a, the tangent line approximation to $f(x)$ is

$$f(x) \approx f'(a)(x - a) + f(a).$$

The expression $f'(a)(x - a) + f(a)$ is called the *local linearization* of f near $x = a$. Note that we are thinking of a as fixed, so that $f'(a)$ and $f(a)$ are constant.

The *error*, $E(x)$, in the approximation is defined by

$$E(x) = f(x) - f(a) - f'(a)(x - a).$$



Examples:

1. What is the tangent line approximation to e^x near $x = 0$?

2. Find the local linearization of $f(x) = x^2$ near $x = 1$.

3. For x near 0, local linearization gives

$$e^x \approx 1 + x.$$

Using a graph, decide if this is an over- or underestimate, and estimate, to one decimal place, the magnitude of error for $-1 \leq x \leq 1$.

4. (a) Find the tangent line approximation to $\cos x$ at $x = \pi/4$.

   (b) Use a graph to explain how you know that whether the tangent line approximation is an under- or overestimate for $0 \leq x \leq \pi/2$.

5. The equation $e^x + x = 2$ has a solution near $x = 0$. By replacing the left-hand side with its local linearization, find an approximate value for the solution.

6. Suppose $f'(x)$ is a differentiable decreasing function for all x. In each of the following pairs, which number is larger? Provide reasons.

   (a) $f'(5)$ and $f'(6)$.

   (b) $f''(5)$ and 0.

   (c) $f(5 + \Delta x)$ and $f(5) + f'(5)\Delta x$.