Section 4.1: Using First and Second Derivatives

What do derivatives tell us about a function and its graph?

Let us reiterate some of the things we've learned about derivatives and what they tell us about the functions they came from.

- If $f' > 0$ on an interval, then f is increasing on that interval.
- If $f' < 0$ on an interval, then f is decreasing on that interval.
- If $f'' > 0$ on an interval, then f is concave up on that interval.
- If $f'' < 0$ on an interval, then f is concave down on that interval.

Example: Use the first and second derivative to analyze the function $f(x) = x^3 - 9x^2 - 48x + 52$.

Local Maxima and Minima

It will be of obvious importance for us to be able to find the locations where a function momentarily reaches a maximum or minimum value.

Suppose p is in the domain of f:

- f has a *local maximum* at p is $f(p)$ is greater than or equal to the values of f for all points near p.
- f has a *local minimum* at p if $f(p)$ is less than or equal to the values of f for all points near p.

How do we find local maxima and minima?

CRITICAL POINTS: For any function f, a point p in the domain of f where $f'(p) = 0$ or $f'(p)$ is undefined is called a *critical point* of the function. In addition, the point $(p, f(p))$ is called a critical point. A *critical value* of f is the value, $f(p)$, at a critical point.

Examples:

1. Find the critical points of $f(x) = 3x^4 - 4x^3 + 6$ and then classify them as local maxima or local minima.

2. Find the critical points of $f(x) = x^5 - 10x^3 - 8$ and then classify them as local maxima or local minima.

THE FIRST DERIVATIVE TEST: Suppose p is a critical point of a continuous function f. Moving from left to right:

- If f' changes from positive to negative at p, then f has a local maximum at p.
- If f' changes from negative to positive at p, then f has a local minimum at p.

THE SECOND DERIVATIVE TEST:

- If $f'(p) = 0$ and $f''(p) > 0$, then f has a local minimum at p.
- If $f'(p) = 0$ and $f''(p) < 0$, then f has a local maximum at p.
- If $f'(p) = 0$ and $f''(p) = 0$, then the test tells us nothing.

3. Find the critical points of $f(x) = 4xe^{3x}$ and classify them as local maxima or minima.

4. Find the critical points of $f(x) = (x^2 - 4)^7$ and then classify them as local maxima or minima.

5. Find the critical points of $f(x) = \dfrac{x}{x^2+1}$ and classify them as local maxima or minima.

Inflection Points and Concavity

DEFINITION: A point, p, at which the graph of a continuous function f changes concavity is called an *inflection point* of f.

The term inflection point can be used to either refer to a number in the domain of f or a point (ordered pair) on the graph of f. The context of the situation should tell you which we are referring to.

How do we detect an inflection point?

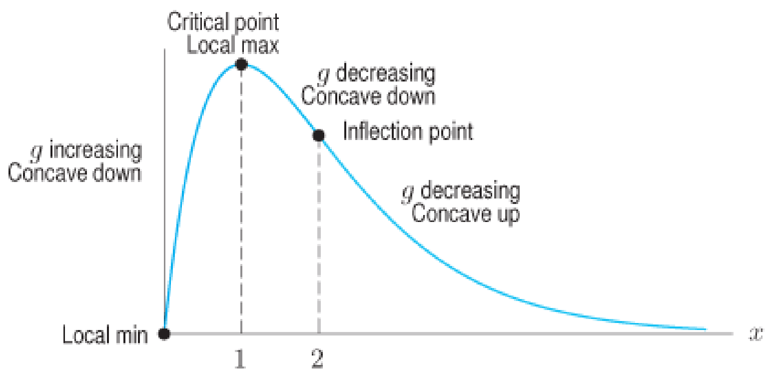
Suppose that f'' is defined on both sides of a point p.

- If f'' is zero or undefined at p, then p is a possible inflection point.
- To test whether p is an inflection point, check whether f'' changes sign at p.

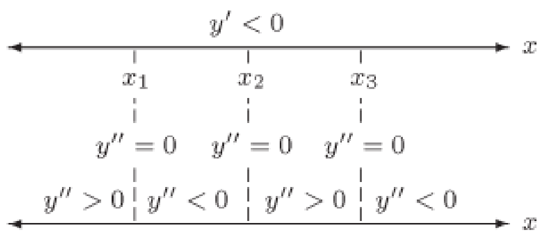
Examples:

6. For $x \geq 0$, find the local maxima and inflection points for $g(x) = xe^{-x}$

Below is the graph of the function from example 6. This general shape can describe a number of important phenomena. For example, if a drug is injected into a patient at time $x = 0$, the amount of the drug present in the patient's bloodstream at time x can be described by a function with this general shape.



7. Using the diagram below, sketch a possible graph of $y = f(x)$, using the given information about the derivatives $y' = f'(x)$ and $y'' = f''(x)$. Assume that the function is defined and continuous for all real numbers x.



8. *Bonus:* Find constants a and b in the function $f(x) = axe^{bx}$ such that f has a local maximum at $(1/3, 1)$.