Section 4.2: Optimization

Global Maxima and Minima

In the last section, we learned how to find critical points and classify them as *local* maxima or minima. In this section, we will learn how to focus our attention in order to find the single greatest (or least) value of a function.

Suppose p is a point in the domain of f:

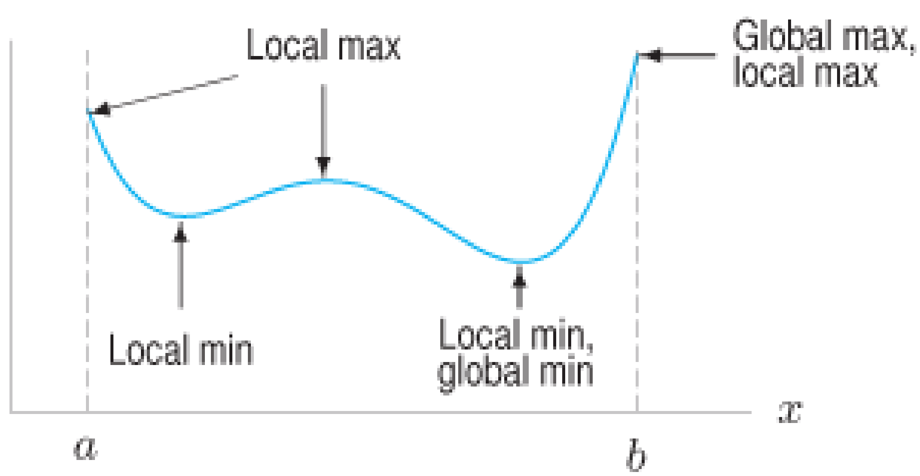
- f has a *global minimum* at p if $f(p)$ is less than or equal to all values of f.
- f has a *global maximum* at p if $f(p)$ is greater than or equal to all values of f.

Global maxima or minima are sometimes called *extrema* or *optimal values*

It turns out that if f is a continuous function, and if we restrict our attention to a closed interval $[a, b]$, then f is guaranteed to obtain both a global maximum and a global minimum.

THEOREM 4.2: THE EXTREME VALUE THEOREM: If f is a continuous function on the closed interval $a \leq x \leq b$, then f has a global maximum and a global minimum on that interval.

Although the extreme value theorem is difficult to prove, it is relatively simple to convince yourself that it must be true just by looking at a picture.



How do we find global maxima and minima?

As we can easily ascertain from just looking at the picture on the previous page and thinking about it a little bit, the only possible places where global extrema might occur on the closed interval $[a, b]$ are at the critical points of f in (a, b) and at the endpoints. Therefore, we can formulate a plan of action:

GLOBAL MAXIMA AND MINIMA ON A CLOSED INTERVAL - TEST THE CANDIDATES: For a continuous function f on a closed interval $a \leq x \leq b$,

- Find the critical points of f in the interval
- Evaluate the function at the critical points and at the endpoints, a and b. The largest value of the function is the global maximum; the smallest value of the function is the global minimum.

Examples:

1. Find the global extrema of $f(x) = x^3 - 3x^2 + 20$ on the interval $-1 \leq x \leq 3$.

2. Find the global extrema of $f(x) = \dfrac{x+1}{x^2+3}$ on the interval $-1 \le x \le 2$.

3. Find the global extrema of $f(x) = xe^{-x^2/2}$ on the interval $-2 \le x \le 2$

How do we find global extrema on an open interval, a half-open interval, or on all real numbers?

> GLOBAL MAXIMA AND MINIMA ON AN OPEN INTERVAL OR ON ALL REAL NUMBERS: For a continuous function f, find the value of f at all the critical points and sketch a graph. Look at the values of x when f approaches the endpoints of the interval, or x approaches $\pm\infty$, as appropriate. If there is only one critical point, look at the sign of f'

4. Find the global extrema of $f(t) = \dfrac{t}{1+t^2}$ on $(-\infty, \infty)$.

4. Find the global extrema of $f(x) = x + 1/x$ for $x > 0$.

5. Find the global extrema of $g(t) = te^{-t}$ for $t \geq 0$.

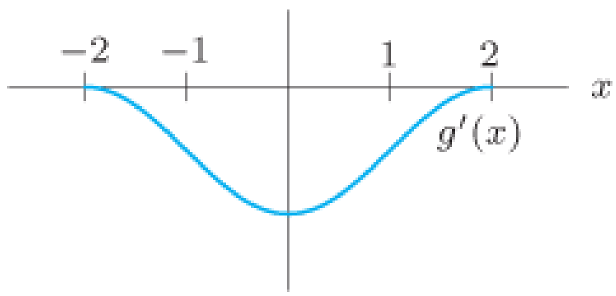
6. Find the best possible bounds for $x^3 - 4x^2 + 4x$ on $[0, 4]$.

7. The potential energy, U, of a particle moving along the x-axis is given by

$$U = b\left(\frac{a^2}{x^2} - \frac{a}{x}\right),$$

where a and b are positive constants and $x > 0$. What value of x maximizes potential energy.

8. The figure below gives the graph of $g'(x)$ for some differentiable function $g(x)$ on the interval $[-2, 2]$.



(a) Describe the behavior of $g(x)$ on this interval

(b) Does the graph of $g(x)$ have any inflection points? If so, give the approximate locations of the inflection points.

(c) Where are the global maxima and minima of g on $[-2, 2]$.