Section 4.7: L'Hopital's Rule, Growth, and Dominance

The whole goal of this section is to find a systematic method for calculating limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)},$$

where $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ (we will also have to consider the case where both function approach ∞ as x approaches a).

Consider, as an example, the limit

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x}.$$

If we attempt to evaluate this limit by direct substitution (by simply plugging in $x = 0$), we get

$$\frac{e^0 - 1}{0} = \frac{0}{0}.$$

A limit of the form $0/0$ is called an *indeterminate form*. Such limits could actually be approaching anything.

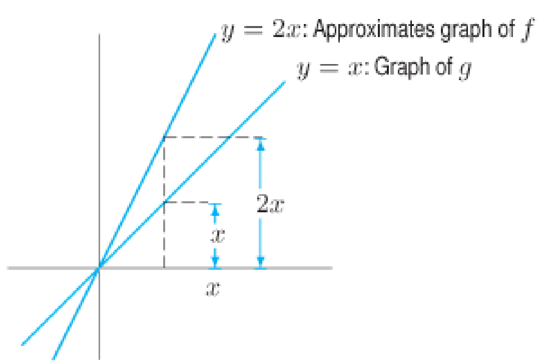
We will calculate the above limit by using *local linearity*. Notice that for values of x very close to zero, we have $e^{2x} - 1 \approx 2x$, and that this approximation gets better the closer x gets to 0. Therefore, we have

$$\lim_{x \to 0} \frac{e^{2x} - 1}{x} = \lim_{x \to 0} \frac{2x}{x} = 2.$$

Notice that what we ended up calculating was

$$\frac{f\prime(0)}{g'(0)}.$$

i.e. the limit of the ration of $f(x)$ over $g(x)$ ended up being the ratio of the slopes of $f(x)$ to $g(x)$ as x approaches 0.



L'Hopital's Rule

L'Hopital's Rule actually gives us a way to generalize this result to limits of the form $0/0$ and ∞/∞.

L'HOPITAL'S RULE: Suppose f and g are differentiable, and that a is any real number or $\pm\infty$. Then if either of the following conditions hold:

- $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$ or
- $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

then it can be shown that

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Examples

1. Find the limit. Use L'Hopital's RUle if it applies.

   (a) $\lim_{x\to 1} \dfrac{x^2+3x-4}{x-1}$

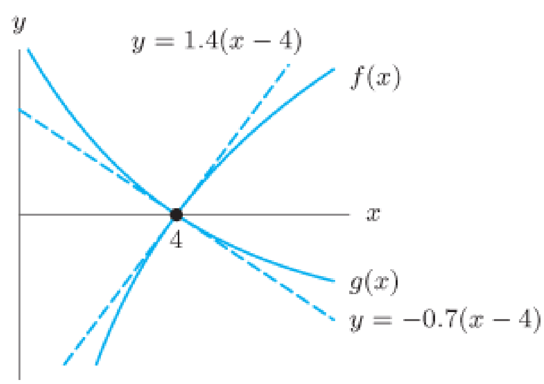
(b) $\lim_{x \to 0} \frac{e^x - 1}{\sin x}$

(c) $\lim_{x \to 1} \frac{\ln x}{x - 1}$

(d) $\lim_{x \to \infty} \frac{(\ln x)^3}{x^2}$

(e) $\lim_{x \to 1} \dfrac{x^a - 1}{x^b - 1}$, where $b \neq 0$.

2. The functions f and g and their tangent lines at $(4, 0)$ are shown in the figure below. Find $\lim_{x \to 4} \dfrac{f(x)}{g(x)}$.



All the Kinds of Indeterminate Forms

Although any indeterminate form, once looked at under the right light, boils down to either

$$\frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty},$$

It is important to recognize all of the different ways that these forms can arise. The following are all of the different types of "indeterminate forms" that we might encounter in this class. Keep in mind that each of these can be realized as one of the two major indeterminate forms in one way or another.

ALTERNATE VERSIONS OF INDETERMINATE FORMS: The following are alternate versions of indeterminate forms.

- $0 \cdot \infty$
- $\infty - \infty$
- 1^{∞}
- 0^{0}
- ∞^{0}

Examples:

2. Describe the form of the limit. Does L'Hopital's Rule apply? Evaluate the limit.

   (a) $\displaystyle\lim_{t\to\infty}\left(\frac{1}{t} - \frac{2}{t^2}\right)$

(b) $\displaystyle \lim_{t \to 0+} \frac{1}{t} - \frac{1}{e^t - 1}$

(c) $\lim_{x\to\infty} (1+x)^{1/x}$

(d) $\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{nt}$

Dominance:

For positive functions f and g, we say that g *dominates* f if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

3. Which function dominates as $x \to \infty$?

   (a) $\ln(x+3)$ or $x^{0.2}$

   (b) x^{10} or $e^{0.1x}$