Section 5.2: The Definite Integral

In this section we learn that we can apply the techniques from section 5.1 to any continuous function $y = f(x)$. The process of calculating the area under a curve is given a special name. It is called the *definite integral.* Let us start with some notational issues.

Sigma Notation:

Suppose $f(x)$ is a continuous function on the interval $a \leq x \leq b$. We divide the interval $[a, b]$ into n equal subdivisions, each with length

$$\Delta x = \frac{b-a}{n}.$$

Let $x_0,\ x_1,\ x_2,\ \ldots,\ x_n$ be the endpoints of the subdivisions. Both the left-hand and right-hand sums can be written more compactly using *sigma*, or summation, notation. The symbol Σ is a Greek capital-S, and stands for "sum".

$$\text{Right} - \text{hand sum} = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^{n} f(x_i)\,\Delta x,$$

and

$$\text{Left} - \text{hand sum} = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\,\Delta x.$$

Both of the sums above are called *Riemann sums* (named after the phenomenal mathematician Bernhard Riemann). The definite integral is then defined by taking the limit as $n \to \infty$ of either the Left-hand sum or the Right-hand sum.

THE DEFINITE INTEGRAL: Suppose f is continuous for $a \leq x \leq b$. The *definite integral* of f from a to b, written

$$\int_a^b f(x)\,dx,$$

is the limit of the left-hand or right-hand sums with n subdivisions of $[a, b]$ as $n \to \infty$. In other words,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\,\Delta x$$

and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Each of these sums is called a *Riemann sum*, f is called the *integrand*, and a and b are called the *limits of integration.*

Examples:

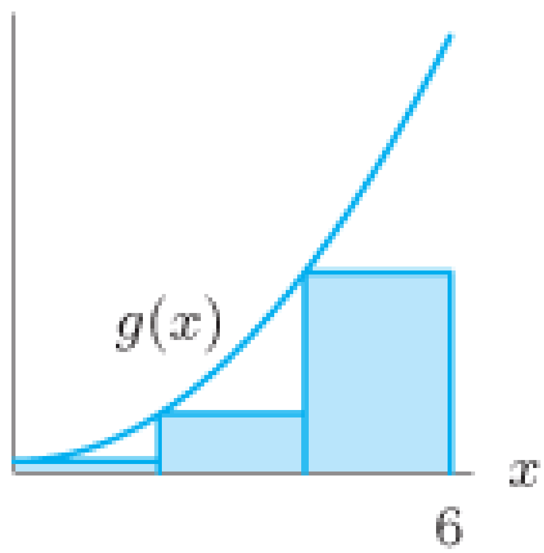
1. Rectangles have been drawn to approximate $\int_0^6 g(x)\,dx$.

   (a) Do the rectangles represent a left-hand or right-hand sum?

   (b) Do the rectangles lead to an upper or a lower estimate?

   (c) What is the value of n?

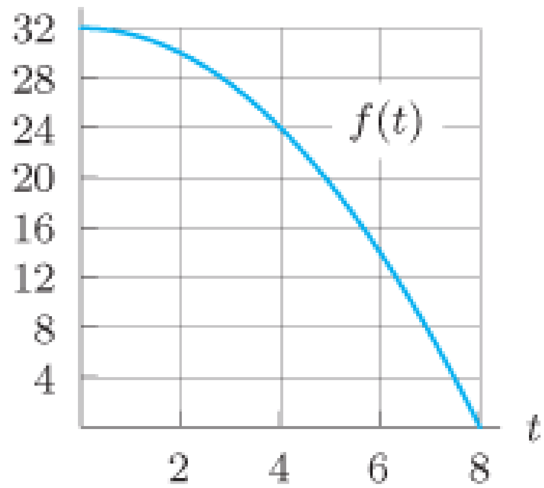
   (d) What is the value of Δx?



2. Draw rectangles representing each of the following Riemann sums for the function f on the interval $0 \leq x \leq 8$. Calculate the value of each sum.

   (a) Left-hand sum with $\Delta t = 2$.

   (b) Right-hand sum with $\Delta t = 2$

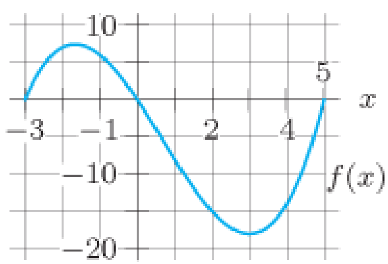


3. Use a computer or a calculator to find the value of $\int_0^3 \ln\left(y^2+1\right)\, dy$

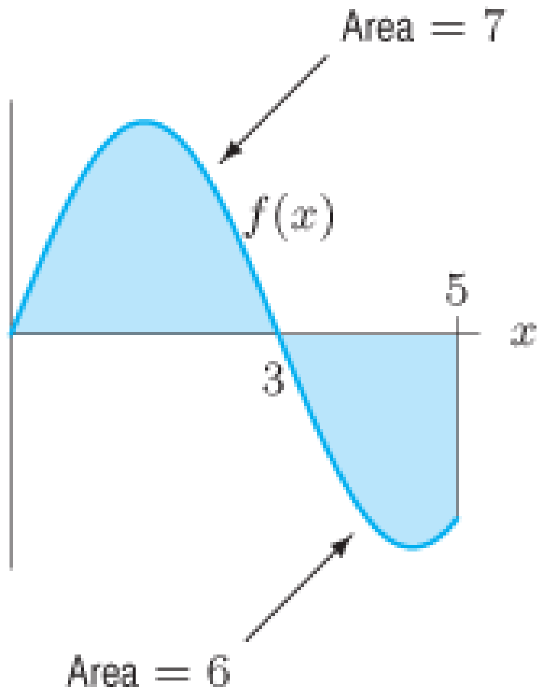
4. Use the table to estimate $\int_0^{12} f(x)\,dx$

x	0	3	6	9	12
$f(x)$	32	22	15	11	9

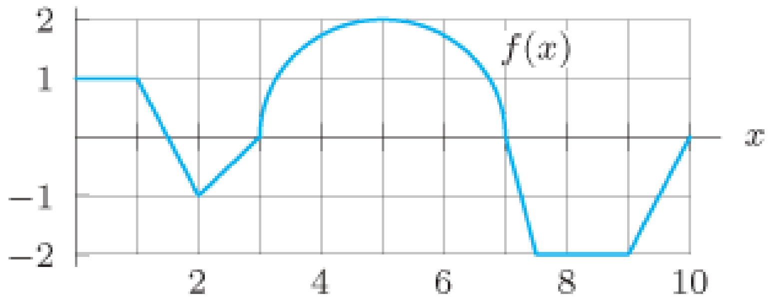
5. Using the figure, estimate $\int_{-3}^{5} f(x)\,dx$



6. What is the area between the graph of $f(x)$ shown below and the x-axis, between $x = 0$ and $x = 5$? What is $\int_0^5 f(x)\,dx$?



7. Use the figure below to find the following values. Note that the figure consists of semicircles and line segments.



(a) $\int_0^2 f(x)\,dx$

(b) $\int_3^7 f(x)\,dx$

(c) $\int_2^7 f(x)\,dx$

(d) $\int_5^8 f(x)\,dx$