Section 5.3: The Fundamental Theorem of Calculus

Question: If $f(x)$ is a continuous function on the interval $[a, b]$, what is the meaning of

$$\int_a^b f(x)\,dx?$$

The notation is built, in a way, to remind us of what the integral means. We are summing up terms of the form $f(x)\,dx$. Think of dx as an "infinitesimal" change in x.

Question: Suppose $s(t)$ represents position, measured in meters, as a function of time, in seconds.

(a) What does $s'(t)$ represent?

(b) What does $\int_0^5 s'(t)\,dt$ represent?

Question: Suppose that $r(t)$ represents the rate at which a population is growing, measured in millions of people per year, where t represents the number of years since 1985. What is the meaning of

$$\int_0^5 r(t)\,dt?$$

We have essentially just deduced one of the most important theorems throughout elementary science, the *fundamental theorem of calculus*. It basically says that the integral of a rate of change of some quantity gives the total change of that quantity.

FUNDAMENTAL THEOREM OF CALCULUS: If f is continuous on the interval $[a, b]$ and if $f(t) = F'(t)$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

We will refer to F as an *antiderivative* of f if $F' = f$.

As stated before, all this theorem is really saying is that integrating the rate of change of a function gives the total change of that function from $t = a$ to $t = b$. Another way of stating the fundamental theorem would be as follows:

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

Question: If we know the units of x and we know the units of $f(x)$, what are the units of $\int_a^b f(x)\,dx$?

Examples:

1. What does $\int_1^3 v(t)\,dt$ represent if $v(t)$ is velocity in m/s and t is time in seconds?

2. What does $\int_{2005}^{2011} f(t)\,dt$ represent if $f(t)$ is the rate at which the world's population is growing in year t, in billions of people per year?

3. Let $f(t) = F'(t)$. Write the integral $\int_{2}^{5} f(t)\,dt$ if $F(t) = 3t^2 + 4t$ and evaluate it using the Fundamental Theorem of Calculus.

4. (a) What is the derivative of $\sin t$?

(b) The velocity of a particle at time t is $v(t) = \cos t$. Use the Fundamental Theorem of Calculus to find the total distance traveled by the particle between $t = 0$ and $t = \pi/2$.

5. (a) If $F(x) = e^{x^2}$, find $F'(x)$.

(b) Find $\int_0^1 2xe^{x^2}\, dx$

6. Oil leaks out of a tanker at a rate of $r = f(t)$ gallons per minute, where t is in minutes. Write a definite integral expressing the total quantity of oil which leaks out of the tanker in the first hour.

7. Let $f(1) = 7$ and $f'(t) = e^{-t^2}$. Use left- and right-hand sums of 5 rectangles each to estimate $f(2)$.