Section 5.4: Theorems About Definite Integrals

THEOREM 5.2: PROPERTIES OF LIMITS OF INTEGRATION: If a, b, and c are any real numbers and f is a continuous function, then

1. $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
2. $\displaystyle\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

THEOREM 5.3: PROPERTIES OF SUMS AND CONSTANT MULTIPLES OF THE INTEGRAND: Let f and g be continuous functions and let c be a constant.

1. $\displaystyle\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
2. $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$

Examples:

1. Given that $\displaystyle\int_a^b f(x)\,dx = 8$, $\displaystyle\int_a^b (f(x))^2\,dx = 12$, $\displaystyle\int_a^b g(t)\,dt = 2$, and $\displaystyle\int_a^b (g(t))^2\,dt = 3$, find

   (a) $\displaystyle\int_a^b cf(z)\,dz$

   (b) $\displaystyle\int_a^b (f(x))^2\,dx - \left(\int_a^b f(x)\,dx\right)^2$

(c) $\displaystyle\int_{a+5}^{b+5} f(x-5)\,dx$

One thing that we can do with definite integrals that turns out to be especially useful is to calculate the area between two curves.

AREA BETWEEN CURVES: If the graph of $f(x)$ lies above the graph of $g(x)$ for $a \le x \le b$, then the area between $f(x)$ and $g(x)$ for $a \le x \le b$ is

$$\int_a^b (f(x) - g(x))\,dx.$$

Examples:

2. Find the area of the regions indicated.

   (a) Between $y = x^2$ and $y = x^3$ for $0 \le x \le 1$

(b) Under $y = 5\ln(2x)$ and above $y = 3$ for $3 \leq x \leq 5$

Interestingly, we can also use the definite integral to give us the *average value* of a function $f(x)$ over an interval $a \leq x \leq b$.

AVERAGE VALUE OF A FUNCTION: The average value of $f(x)$ over the interval $[a, b]$ is given by

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

Note: It isn't difficult to see, using Riemann sums, how this definition of an average coincides with our natural understanding of averages.

Examples:

3. If the average value of a function f on the interval $2 \leq x \leq 5$ is 4, find $\displaystyle\int_2^5 (3f(x) + 2)\,dx$

Finally, we can consider what will happen when we integrate over intervals of the form $[-a, a]$ if f is either an even function or an odd function.

Question: Using symmetry, give formulas for the definite integral of a function $f(x)$ over an interval of the form $[-a, a]$ in the case that f is odd and f is even.

Examples:

4. If $f(x)$ is even and $\displaystyle\int_{-2}^{2} (f(x) - 3)\, dx = 8$, find $\displaystyle\int_{0}^{2} f(x)\, dx$

5. Without any computation, find $\displaystyle\int_{-\pi/4}^{\pi/4} x^3 \cos(x^2)\, dx$

6. Using geometry, construct a formula for $\int_a^b x\,dx$

7. Using the result from problem 6, find $\int_1^3 5x\,dx$