Section 6.2: Constructing Antiderivatives Analytically

In this section, we will begin to construct algebraic expressions for antiderivatives of functions. Recall that F is an *antiderivative* of f if $F' = f$ We will begin by going over some preliminary facts.

If F and G are both antiderivatives of f on an interval, then $G(x) = F(x) + C$.

The Indefinite Integral:

Since antiderivatives are only unique up to a constant, they all have the form $F(x) + C$. We will introduce a special notation for the antiderivative which is derived from the notation for a definite integral. It's called an *indefinite integral*, and the notation is just the integral notation without the limits of integration.

THE INDEFINITE INTEGRAL: If $F(x)$ is any antiderivative of $f(x)$, then the *indefinite integral* of f is defined as

$$\int f(x)\,dx = F(x) + C.$$

The constant C is referred to as the *constant of integration*.

It is important to note the difference between $\int_a^b f(x)\,dx$ and $\int f(x)\,dx$ The definite integral is a *number*, whereas the indefinite integral is a family of functions.

What is the antiderivative of a constant $f(x) = k$?

If k is any constant, then $\frac{d}{dx}kx = k$. Thus we arive at the following result.

If k is any constant,

$$\int k\,dx = kx + C.$$

Now let us just figure out some basic antiderivatives just by thinking about it a little bit.

Examples:

1. Find the following indefinite integrals.

   (a) $\int x\,dx$

(b) $\int x^2 \, dx$

(c) $\int x^3 \, dx$

(d) $\int x^4 \, dx$

(e) $\int x^5 \, dx$

We arrive at the following conclusion.

THE POWER RULE: For any real number $n \neq -1$, we have

$$\int x^n \, dx = \frac{x^{n+1}}{n+1}$$

The following are some more antiderivatives of the most common types of functions we encounter. These are all easily checked.

$$\int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

And finally, here are some useful properties which follow directly from the properties for differentiation.

PROPERTIES OF ANTIDERIVATIVES: SUMS AND CONSTANT MULTIPLES:

1. $\displaystyle\int (f(x) \pm g(x)) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$

2. $\displaystyle\int cf(x) \, dx = c \int f(x) \, dx$

Examples:

2. Find the following indefinite integrals.

   (a) $\displaystyle\int \left(4t + \frac{1}{t}\right)\, dt$

   (b) $\displaystyle\int (4e^x - 2\sin x)\, dx$

   (c) $\displaystyle\int \left(\frac{3}{t} - \frac{2}{t^2}\right)\, dt$

(d) $\int t^3(t^2+1)\,dt$

(e) $\int \frac{x+1}{x}\,dx$

3. Find an antiderivative $F(x)$ with $F'(x) = f(x)$ and $F(0) = 0$.

   (a) $f(x) = 2 + 4x + 5x^2$

(b) $f(x) = \sin x$

Computing Definite Integrals Using Antiderivatives:

The fundamental theorem of calculus makes it easy for us to compute definite integrals once we know how to find antiderivatives. We introduce the following notation to make the calculations flow more smoothly.

If $F(x)$ is any antiderivative of a continuous function $f(x)$ on $[a, b]$, we have

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

4. Evaluate the definite integral exactly.

(a) $\displaystyle\int_1^3 \frac{1}{t}\,dt$

(b) $\int_0^1 2e^x \, dx$

(c) $\int_2^5 (x^3 - \pi x^2) \, dx$

(d) $\int_0^{\pi/4} (\sin t + \cos t) \, dt$