Section 6.4: The Second Fundamental Theorem of Calculus

Exercise: Consider the statement of the Fundamental Theorem of Calculus (FTC):

$$F(b) - F(a) = \int_a^b f(t)\,dt.$$

Use the FTC, as written above, to write down an expression for $F(x)$, given that $F(0) = 0$ [Hint: write down the FTC over the interval $[0, x]$.

The above is referred to as the *Construction Theorem*, or the *Second Fundamental Theorem of Calculus.*

THEOREM 6.2: CONSTRUCTION THEOREM FOR ANTIDERIVATIVES: If f is a continuous function on an interval, and if a is any number in that interval, then the function F defined on the interval defined as follows is an antiderivative of f:

$$F(x) = \int_a^x f(t)\,dt.$$

The construction theorem gives us a useful way of representing any antiderivative $F(x)$ of a function $f(x)$, and we can use it to represent an antiderivative even if it is impossible to find a closed form antiderivative of a specific function.

For example, it can be shown that the function $f(x) = e^{-x^2}$ has no closed-form antiderivative. However, we can still use the notation from the construction theorem to refer to an antiderivative of $f(x)$:

$$F(x) = \int_0^x e^{-t^2}\,dt.$$

If we can get good numerical estimates for the areas given by the equation above for different values of x, then we can obtain useful information about the numerical values of its antiderivative.

Note that if $f(x)$ is continuous on an interval $[a, b]$, and if x is any point in the interval (a, b), the construction theorem gives us

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Examples:

1. For $x = 0$, 0.5, 1.0, 1.5, and 2.0, make a table of values for $I(x) = \displaystyle\int_0^x \sqrt{t^4+1}\,dt$.

2. Write an expression for the function, $f(x)$, with the given properties.

   (a) $f'(x) = \sin(x^2)$ and $f(0) = 7$.

   (b) $f'(x) = \mathrm{Si}(x)$ and $f(1) = 5$. (Note that $\mathrm{Si}(x)$ is defined as $\mathrm{Si}(x) = \displaystyle\int_0^x \frac{\sin t}{t}\,dt$)

3. Find the indicated derivatives:

(a) $\dfrac{d}{dt}\displaystyle\int_4^x \sin(\sqrt{x})\,dx$

(b) $\dfrac{d}{dx}\displaystyle\int_2^x \ln(t^2+1)\,dt$

(c) $\dfrac{d}{dx}\mathrm{Si}(x^2)$

4. Let $F(x) = \int_0^x \sin(t^2)\,dt$. Does $F(x)$ have a maximum value for $0 \le x \le 2.5$? If so, at what value of x does it occur, and approximately what is the maximum value?

5. Use the chain rule to calculate the following derivatives:

   (a) $\dfrac{d}{dt}\displaystyle\int_1^{\sin t} \cos(x^2)\,dx$

(b) $\dfrac{d}{dx}\displaystyle\int_{-x}^{x^2} e^{t^2}\,dt$

(c) $\dfrac{d}{dx}(\mathrm{erf}(\sqrt{x}))$, where the *error function* is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

(d) $\dfrac{d}{dx}\displaystyle\int_{x}^{x^3} e^{-t^2}\,dt$