Study Guide 3

Math 122B Instructor: M. Gilbert

1. Find the global maximum and minimum for the following functions on the closed interval:

   (a) $f(x) = x^3 - 3x^2 + 20$ on the interval $[-1, 3]$.

   (b) $f(x) = xe^{-x^2/2}$ on the interval $[-2, 2]$.

(c) $f(x) = \dfrac{x+1}{x^2+3}$ on the interval $[-1, 2]$.

(d) $f(x) = x - 2\ln(x+1)$ on the interval $[0, 2]$.

2. Find the exact global maximum and minimum (if they exist) for the function $f(x) = x - \ln x$ for $x > 0$.

3. Find the exact global maximum and minimum (if they exist) for the function $g(t) = ate^{-bt}$ for $t \geq 0$, if a, b are positive constants.

4. Which point on the curve $y = \sqrt{1-x}$ is closest to the origin?

5. The perimeter of a rectangle is 64 cm. Find the lengths of the sides of the rectangle which give the maximum area.

6. A rectangle has one side on the x-axis, and two vertices (equal distance from the y-axis) placed on the curve

$$y = \frac{1}{1+x^2}.$$

Find the vertices of the rectangle with maximum area.

7. A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can he do this so as to minimize the cost of the fence?

8. A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

9. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and sides and a 2-inch margin at the top. What dimensions will give the largest possible printed area?

10. A cone-shaped paper drinking cup is to be made to hold 27 cm^3 of water. Find the height and the radius of the cup that will use the smallest amount of paper. [Hint: The surface area of the lateral portion of a right-circular cone is given by $A = \pi r\sqrt{r^2 + h^2}$.]

11. A right circular cylinder is inscribed in a sphere of radius R. Find the largest possible volume of such a cylinder.

12. A right circular cylinder is inscribed in a cone with height H and base radius R. Find the largest possible volume of such a cylinder.

13. A norman window has the shape of a rectangle surmounted by a semicircle. (Thus, the diameter of the semicircle is equal to the width of the rectangle.) If the perimeter of the window is to be 30 ft., find the dimensions of the window so that the greatest possible amount of light is admitted.

14. A rectangular storage container with an open top is to have a volume of 10 m^3. The length of its base is twice the width. Material for the base costs \$10 per square meter. Material for the sides costs \$6 per square meter. Find the cost of materials for the cheapest such container.

15. In the following exercises, investigate the given two parameter family of functions. Assume that a and b are positive.

    (a) Graph $f(x)$ using $b = 1$ and three different values for a.
    (b) Graph $f(x)$ using $a = 1$ and three different values for b.
    (c) In the graphs of parts (a) and (b), how do the critical points of f appear to move as a increases? As b increases?
    (d) Find a formula for the x-coordinates of the critical point(s) of f in terms of a and b.

    (A) $f(x) = (x - a)^2 + b$

    (B) $f(x) = x^3 - ax^2 + b$

(C) $f(x) = ax(x-b)^2$

(D) $f(x) = \dfrac{ax}{x^2+b}$

(E) $f(x) = \sqrt{b - (x-a)^2}$

(F) $f(x) = \dfrac{a}{x} + bx$ for $x > 0$.

16. Consider the family of functions $y = be^{-(x-a)^2/2}$. Find values of a and b so that y has its maximum at the point $(0, 3)$.

17. Find values of A and B so that the function $y = \dfrac{a}{1 + be^{-t}}$ has a y-intercept of 2 and an inflection point at $t = 1$.

18. If a snowball melts so that its surface area decreases at a rate of 1 $\mathrm{cm}^2/\mathrm{min}$, find the rate at which the diameter decreases when the diameter is 10 cm.

19. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft away from the pole?

20. The altitidue of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

21. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top, and has a height of 50 cm. If the trough is being filled with water at the rate of 0.2 m^3/min, how fast is the water level rising when the water is 30 cm deep?

22. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $pi/3$.

23. A ladder 10 ft. long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 ft/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/4$?

24. Determine whether the limit exists, and if possible, evaluate it.

(a) $\lim\limits_{t \to \pi} \dfrac{\sin^2 t}{t - \pi}$.

(b) $\lim\limits_{x \to 0+} x \ln x$.

(c) $\lim\limits_{x \to 0} \dfrac{1 - \cosh(3x)}{x}$

(d) $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x} \right)$

(e) $\lim_{t \to 0} \frac{\sin^2(At)}{\cos(At) - 1}$, where $A \neq 0$.

(f) $\lim_{x \to 0+} x^a \ln x$, where a is a positive constant.

25. Determine which function dominates as $x \to \infty$.

   (a) x^5 and $0.1x^7$.

   (b) $\ln(x+3)$ and $x^{0.2}$.

   (c) $0.01x^3$ and $50x^2$.

   (d) x^{1000} and $e^{0.1x}$