Section 12.1: Functions of Two Variables

We should all be comfortable with functions of the form $y = f(x)$, where x is the *independent variable* (input) and y is the *dependent variable* (output). In general, the independent variable and the dependent variable can be a multitude of objects. However, we are going to concern ourselves primarily with the situation in which the independent and dependent variables are each elements of the real numbers.

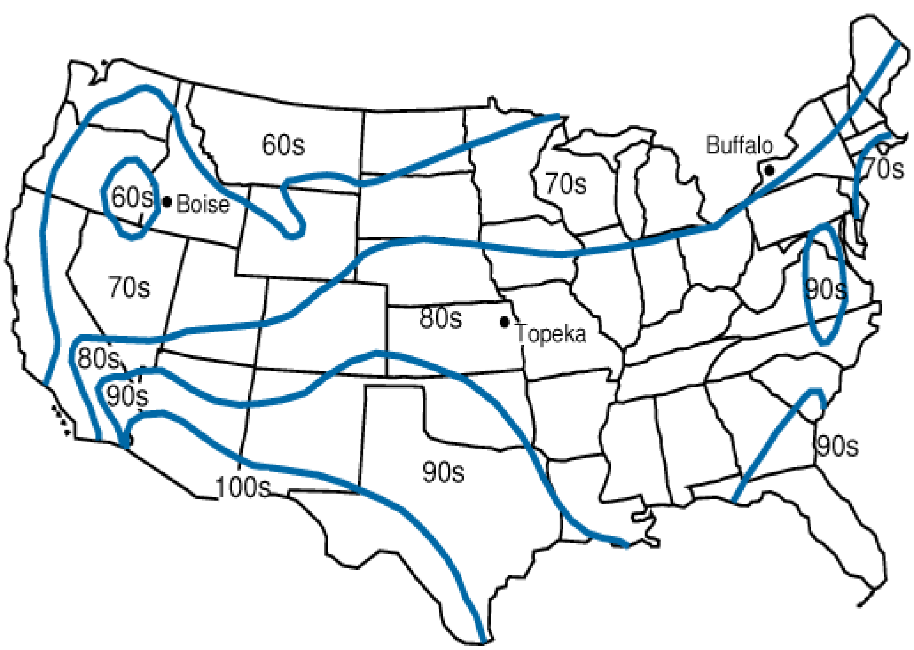
It is not difficult to imagine a situation in which a function could require more than one independent variable. For example, knowing the temperature of a planar plat at a point (x, y) on the plane would require us to know both of the coordinates x and y. We might write such a function as

$$H = f(x, y).$$

In general, when dealing with functions of two variables, we will write $z = f(x, y)$ to represent a function with independent variables x and y, and dependent variable z. There are three natural ways to represent such functions: graphically, numerically (using a table), and algebraically.

Graphical Example:

Consider the image shown below of the United States. The image conveys the predicted high temperature, T, as a function of the point (x, y), measured in degrees Fahrenheit (F°). Each of the blue curves is called an *isotherm*, and represents a curve of *constant temperature*.



Numerical Example

The temperature adjusted for wind chill is a temperature which tells you how cold it feels, as a result of the combination of wind speed and temperature. See the table below

Temperature (�F)

Wind Speed (mph)	**35**	**30**	**25**	**20**	**15**	**10**	**5**	**0**
5	31	25	19	13	7	1	−5	−11
10	27	21	15	9	3	−4	−10	−16
15	25	19	13	6	0	−7	−13	−19
20	24	17	11	4	−2	−9	−15	−22
25	23	16	9	3	−4	−11	−17	−24

(a) If the temperature is $0°$ F and the wind speed is 15 mph, how cold does it feel?

(b) If the temperature is $35°$ F, what wind speed makes it feel like $24°$ F?

(c) If the temperature is $25°$ F, what wind speed makes it feel like $12°$ F?

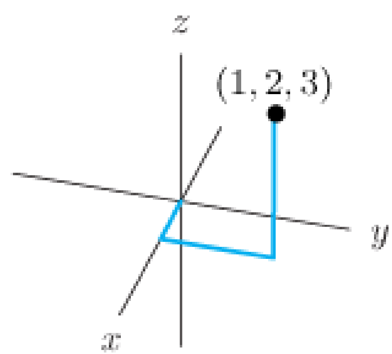
(d) If the wind is blowing at 20 mph, what temperature feels like $0°$ F?

Algebraic Example:

Write a formula $V = f(r, h)$ for the volume of a cone.

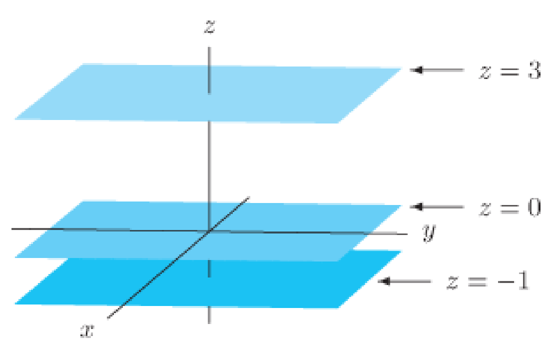
3-Space ($\mathbb{R}^3$)

Since we covered chapter 13 before this section, we have already become somewhat aquainted with $\mathbb{R}^3$. The image below shows how one might graph the point $(1, 2, 3)$ in $\mathbb{R}^3$.



Graphing Equations in $\mathbb{R}^3$

The simplest types algebraic formulas that we can graph are those for the planes which are parallel to the coordinate planes (the xy, xz, and yz-planes). For example, any plane that is parallel to the xy-plane will have an equation of the form $z = k$, where k is any constant.



Question: What are the general equations for planes parallel to the xz and yz-planes, respectively?

Examples:

1. Which of the points $A = (1.3, -2.7, 0)$, $B = (0.9, 0, 3.2)$, and $C = (2.5, 0.1, -0.3)$ is closest to the yz-plane? Which one lies on the xz-plane? Which one is furthest from the yz-plane?

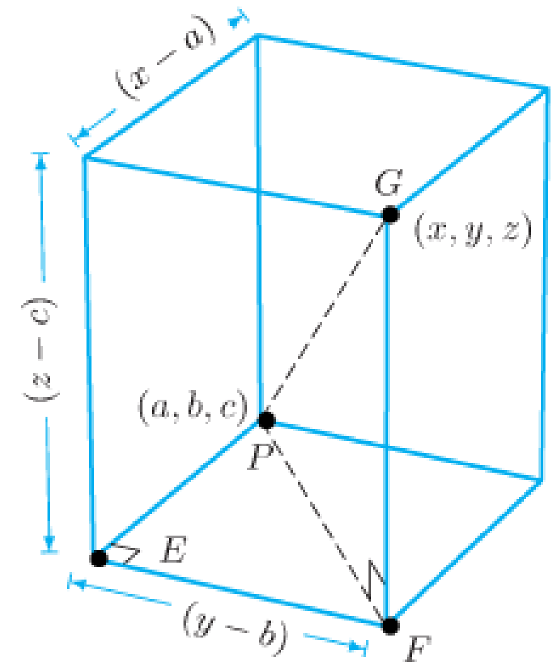
2. What is the distance from the point (x, y, z) to the x-axis, y-axis, and z-axis?

Distance Between Two Points

Recall that in $\mathbb{R}^2$, the distance between two points (a, b) and (x, y) is given by applying the Pythagorean Theorem:

$$\text{Distance} = \sqrt{(x-a)^2 + (y-b)^2}.$$

Using this fact, along with the figure below, come up with a formula describing the distance between the points (a, b, c) and (x, y, z) in $\mathbb{R}^3$.



The distance between the points (a, b, c) and (x, y, z) in $\mathbb{R}^3$ is

$$\text{Distance} = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

Examples:

3. Find an equation of the sphere of radius 2 centered at $(1, 0, 0)$.

4. (a) Find the equations of the circles (if any) where the sphere $(x-1)^2+(y+3)^2+(z-2)^2 = 4$ intersects each one of the coordinate planes.

   (b) Find the points (if any) where the sphere intersects each of the coordinate axes.

5. Describe the set of all points whose distance from the yz-plane is equal to the distance from the x-axis.