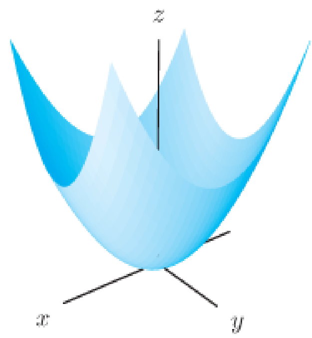
Section 12.2: Graphs and Surfaces

Visualizing a Function of Two Variables Using a Graph

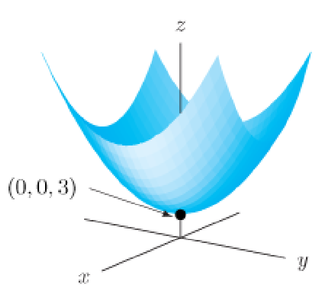
The graph of a two-variable function f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$. In general, the graph of a two-variable function is a two-dimensional surface sitting in $\mathbb{R}^3$.



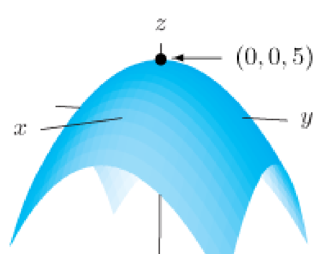
The figure above shows the graph of the two-variable function $f(x, y) = x^2 + y^2$ for $-3 \le x \le 3$, $-3 \le y \le 3$.

New Graphs from Old

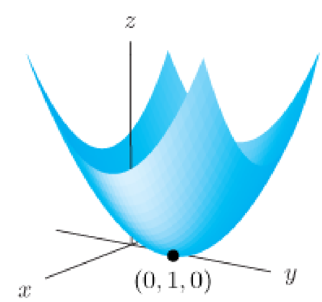
In the same way as with functions of a single variable, it is possible to create new graphs of two-variable functions through translations, reflections, stretches, and compressions. Pictured below are some examples of graphs which were obtained from the graph of $f(x, y) = x^2 + y^2$



Graph of $g(x, y) = x^2 + y^2 + 3$



Graph of $h(x, y) = 5 - x^2 - y^2$

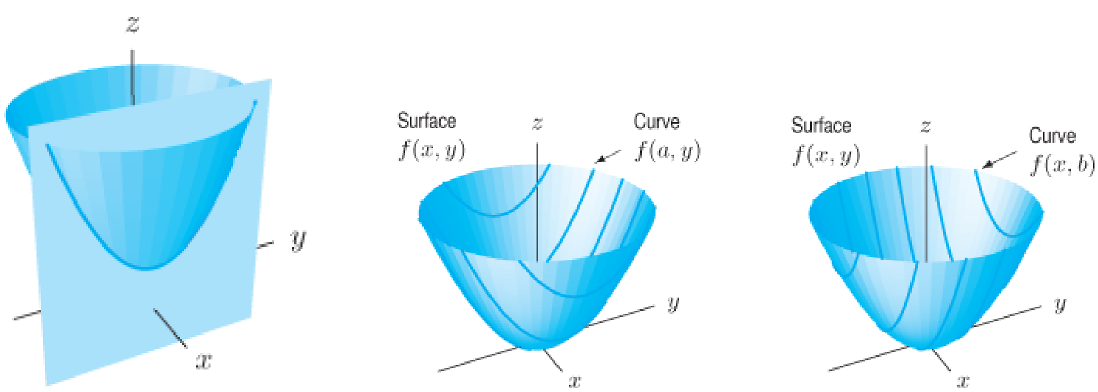


Graph of $k(x, y) = x^2 + (y - 1)^2$

Question: Describe the graph of $g(x,y) = e^{-(x^2+y^2)}$. What kind of symmetry does it have?

Cross-Sections of the Graph of a Function

For a function $f(x,y)$, the function we get by holding x fixed and letting y vary is called a *cross-section* of f with x fixed. The graph of the cross-section of $f(x,y)$ with $x = c$ is the curve, or cross-section, we get by intersecting the graph of $f(x,y)$ with the plane $x = c$. We define a cross-section of f with y fixed similarly.



Examples:

1. Consider the function f given by $f(x, y) = y^3 + xy$. Draw graphs of cross sections with:

   (a) x fixed at $x = -1$, $x = 0$, and $x = 1$.

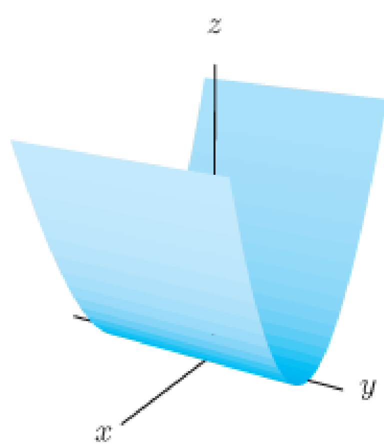
   (b) y fixed at $y = -1$, $y = 0$, and $y = 1$.

Question: What is the shape of the graph of a *linear* function of the form $f(x, y) = mx + ny + d$?

When one Variable is Missing: Cylinders

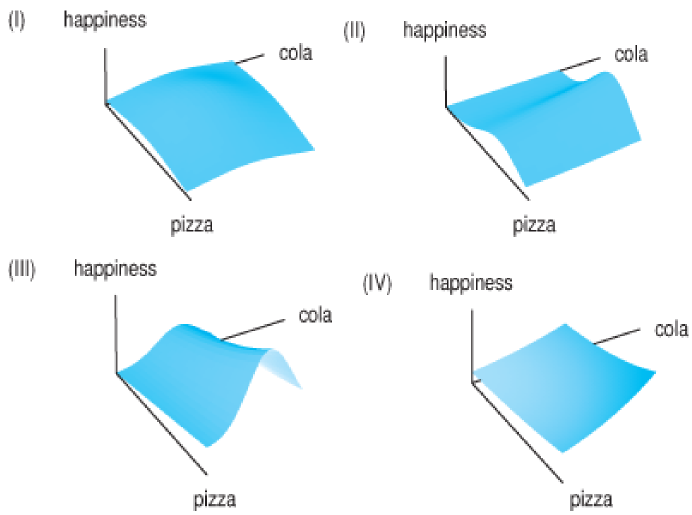
How would you go about visualizing the graph of the function $z = x^2$ in $\mathbb{R}^3$? It is very easy to understand that if this equation were restricted to the xz-plane, we would simply get a pretty standard parabola.

In fact, each of the cross-sections of z with y fixed will be the exact same parabola. Therefore, by letting y vary, this parabola will sweep out a very obvious surface:



Examples:

2. You like pizza and you like cola. Which of the graphs in the figure below represents your happiness as a function of how many pizzas and how much cola you've had if



(a) there is no such thing as too many pizzas and too much cola?

(b) there is such a thing as too many pizzas and too much cola?

(c) there is such a thing as too much cola but no such thing as too many pizzas?

3. By setting one variable constant, find a plane that intersects the graph of $z = 4x^2 - y^2 + 1$ in a

   (a) parabola opening upward

   (b) parabola opening downward

   (c) pair of intersecting straight lines