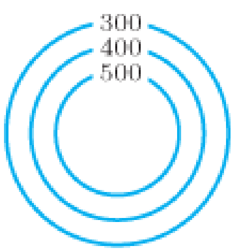
Section 12.3: Contour Diagrams

The graph of a two-variable function in $\mathbb{R}^3$ can provide us with a lot of insight into the behavior of the function. However, it is often difficult to distinguish many important aspects of the function due to the nature of having to draw the graph on a two-dimensional plane. Therefore, we often use *contour diagrams* to give a visual representation of a two-variable function. A contour diagram is simply a graph on the xy-plane that shows curves of equal height for a two-variable function $z = f(x, y)$.

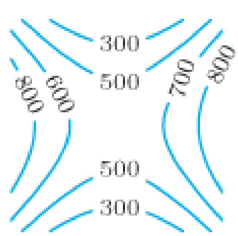
Question: What are some examples of contour diagrams that you can think of?

Topographical Maps

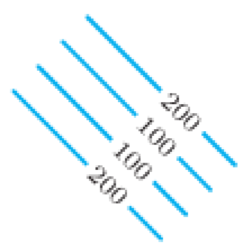
Topographical maps are probably the most common types of contour diagrams that people regularly encounter. Below are some examples of different types of topographical regions and the contours that they give rise to. You should take note of the fact that the elevation numbers are just as important as the curves. We usually graph contour diagrams for equally spaced values of z.



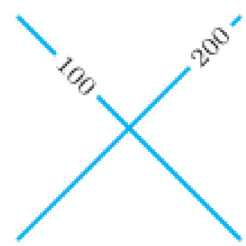
Mountain peak



Pass between mountains



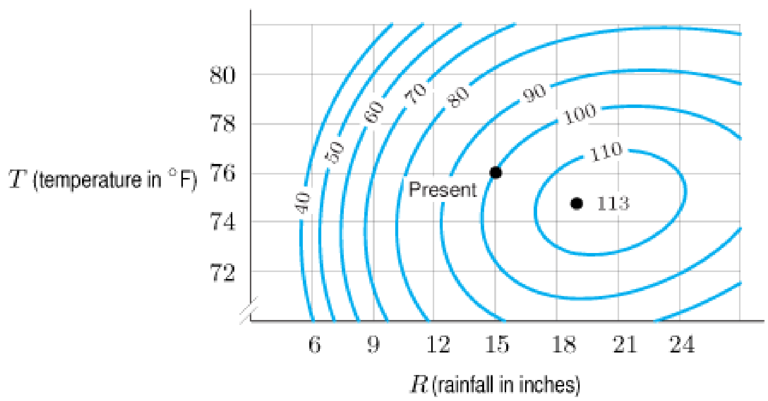
Long valley



Impossible contour lines

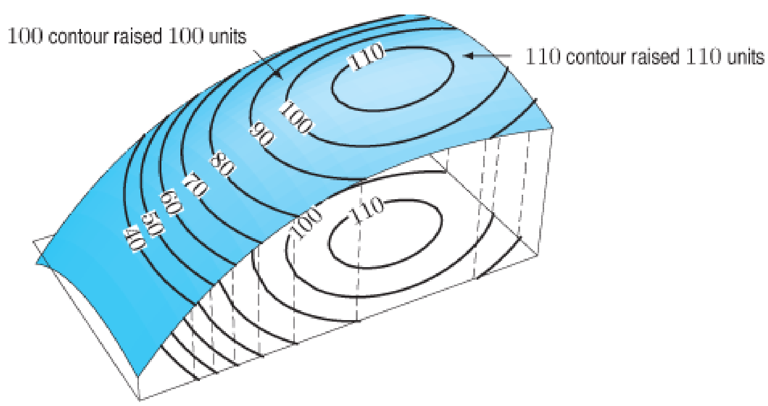
Corn Production

The contour map below shows the effect of weather on US corn production. Specifically, it gives the contour lines for the production function $C = f(R, T)$, where C is corn production, R is total rainfall, in inches, and T is the average temperature, in degrees Fahrenheit.



Use the figure above to estimate $f(18, 78)$ and $f(12, 76)$, and interpret.

The figure below shows how the contour diagram from the corn production example is related to the graph of the corn production function $C = f(R, T)$.



Contour lines, or *level curves*, are obtained from a surface by slicing it with horizontal planes. A contour diagram is a collection of level curves labeled with function values.

Finding Contours Algebraically

If we have a formula for a function $z = f(x, y)$, then we can find the equations for the contours easily. Each contour is obtained by slicing the surface with the horizontal plane $z = c$, so the equation for the contour at height c is simply

$$f(x, y) = c.$$

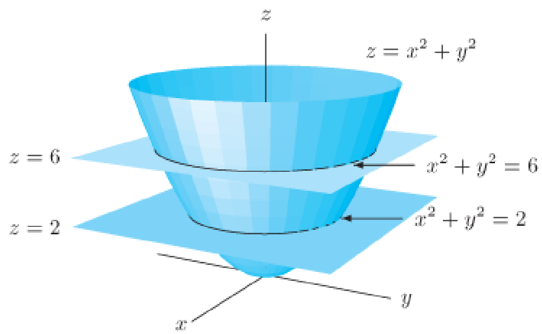
Examples:

1. Sketch a contour diagram for the function with at least four labeled contours.

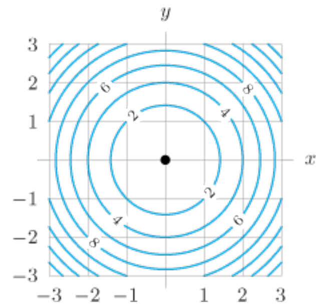
   (a) $f(x, y) = x^2 - y^2$

   (b) $f(x, y) = \cos\sqrt{x^2 + y^2}$.

The two images below show the graph of $f(x, y) = x^2 + y^2$ being sliced by horizontal planes, and the accompanying contour diagram.



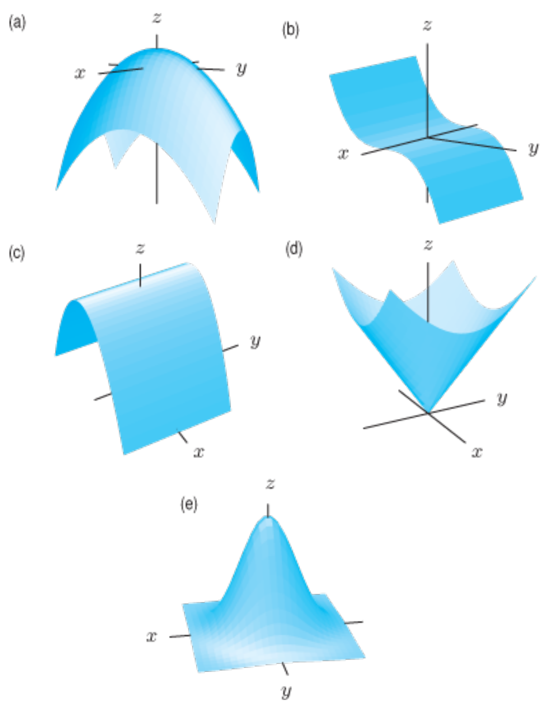
The graph of $f(x, y) = x^2 + y^2$

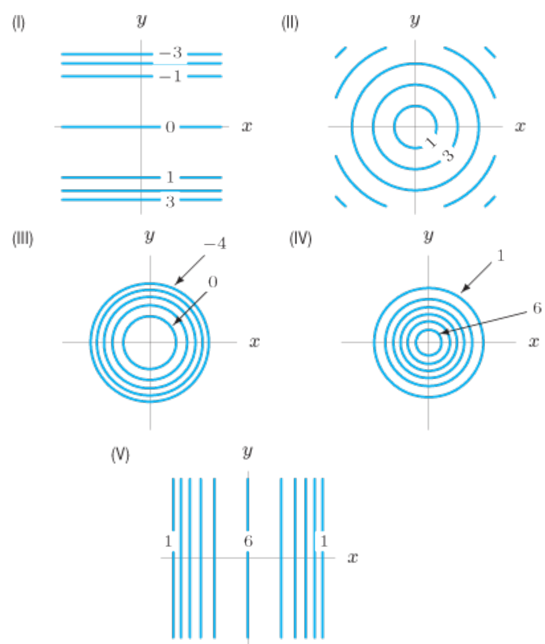


Contour diagram for $f(x, y) = x^2 + y^2$

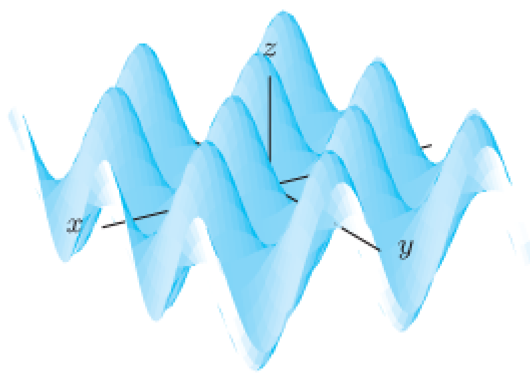
Examples:

2. Match the surfaces (a)-(e) in the figure below with their respective contour diagrams





3. The figure below shows a graph of $f(x, y) = (\sin x)(\cos y)$ for $-2\pi \leq x \leq 2\pi$, $-2\pi \leq x \leq 2\pi$. Use the surface $z = 1/2$ to sketch the contour $f(x, y) = 12$.



Using Contour Diagrams: The Cobb-Douglas Production Function

The Cobb-Douglas function is a function for modeling production, P, as a function of the number of workers, N, and the total value, V, of your equipment. The general form is

$$P = f(N, V) = cN^{\alpha}V^{\beta},$$

where c, α, and β are positive constants such that $0 < \alpha < 1$ and $0 < \beta < 1$.

The figure below shows possible contour lines for a Cobb-Douglas production model